(2-3)

$$I(s) = \sum_{m=0}^{2g-2} \sum_{V \in \mathcal{M}_{X,n}(mn)} \frac{q^{h^0(X,V)}}{\# \mathrm{Aut}\, V} (q^{-s})^{\chi(X,V)} \qquad V = K_X \otimes W^* \qquad \mathrm{Aut}(V) \simeq \mathrm{Aut}(W)$$

$$= \sum_{m=0}^{2g-2} \sum_{[K_X \otimes W^*] \in \mathcal{M}_{X,n}(mn)} \frac{q^{h^0(X, K_X \otimes W^*) = h^1(X,W) = h^0(W) - \chi(W)\ \deg W}}{\# \mathrm{Aut}(W)} (q^{-s})^{-\chi(X,W)}$$

$$= \sum_{m=2g-2}^{0} \sum_{[W] \in \mathcal{M}_{X,n}(mn)} \frac{q^{h^0(X,W)}}{\# \mathrm{Aut}(W)} \cdot (q^{s-1})^{\chi(X,W)} = I(1-s).$$

$\Rightarrow$ F.E

More careful calculations lead to:

$$\hat{\zeta}_{X,n}(s) = \sum_{m=0}^{(g-1)-1} \alpha_{X,n}(mn) \left[\left(\frac{1}{T}\right)^{(g-1)-m} + (QT)^{(g-1)-m} \right]$$

$$+ \alpha_{X,n}(n(g-1)) + \beta_{X,n}(0) \cdot \frac{QT^g}{(1-T)(1-QT)} \quad \Leftarrow -\mathrm{I} + \mathrm{IV}$$

Hence. Def α & β invariants [Non-Abelian Invariants]

$$\alpha_{X,n}(d) := \sum_{[V] \in \mathcal{M}_{X,n}(d)} \frac{q^{h^0(X,V)} - 1}{\# \mathrm{Aut}(V)}. \quad \Leftarrow$$

[Invariants reflects of arithmetic of Brill-Noether Loci

(2) Geometric Interpretation (Moduli One)

Moduli stacks $\mathcal{M}_{X,n}(0),\ \mathcal{M}_{X,n}(n),\ \mathcal{M}_{X,n}(2n),\ \cdots,\ \mathcal{M}_{X,n}((2g-2)n)$

$\mathcal{M}_{X,n}((g-1)n)$

duality.

$[\mathcal{M}_{X,n}^d$

Brill-Noether Loci:

$$\mathcal{M}_{X,n}^{\geq i}(d) := \{ V : h^0(X,V) \geq i \}$$

$$\mathcal{M}_{X,n}^{= i}(d) := \mathcal{M}_{X,n}^{\geq i}(d) \setminus \mathcal{M}_{X,n}^{\geq (i+1)}(d) \quad \swarrow$$

locally constant function on $\mathcal{M}_{X,n}^{=i}(d)$

Ex $\mathrm{Pic}^{g-1}(X) = \mathcal{M}_{X,1}(g-1)$

$\Rightarrow \chi(X,L) = 0$ i.e. $h^0(X,L) = h^1(X,L)$

$\mathcal{M}_{X,1}^{\geq 0}(g-1) = \mathrm{Pic}^{g-1}(X)$, total space, $\mathcal{M}_{X,1}^{\geq 1}(g-1) = \{ L \in \mathrm{Pic}^{g-1}(X) : h^0(L) \geq 1 \}$ theta divisor of $\mathrm{Pic}^{g-1}(X)$

$\mathcal{M}_{X,1}^{\geq 2}(g-1) = \{ L : h^0(L) \geq 2 \} = \mathrm{Sing}(\Theta)$ In general / determinantal varieties $\mathcal{M}_{X,n}^{\geq i+1} = \mathrm{Sing}\, \mathcal{M}_{X,n}^{\geq i}$.