Lemma. (i) Riemann-Roch. $h^0(X,V) - h^1(X,V) = \deg(V) - n(g-1)$ $\quad n := \mathrm{rk}(V)$.

$\deg(V) = \deg(\det V)$ line bdl. L, s: rational section.

$\mathrm{div}(s) = \sum_p \mathrm{ord}_p(s)\cdot p$.

$\deg(s) := \deg(\mathrm{div}(s))$

well-defined.

$\mathrm{div}(s) \sim \mathrm{div}(s') \iff s' = f\cdot s$

$\mathrm{div}(s') = \mathrm{div}(s) + \mathrm{div}(f)$

$\deg(s) = \deg(s')$

(ii) (Vanishing thm) $\deg(V) > (2g-2)\cdot n$, V: s.stable $\Rightarrow h^1(X,V) = 0$.

$\because)$ V: s.stable $\Rightarrow K_X \otimes V^*$: semi-stable

$\deg V > (2g-2)n \Rightarrow \deg(K_X \otimes V^*) = \deg K_X \cdot \mathrm{rk}(V^*) + \mathrm{rk}(K_X)\cdot \deg V^*$

$< (2g-2)\cdot n + 1\cdot[-(2g-2)\cdot n] = 0$.

if $H^0(X, K_X\otimes V^*) \neq 0 \Rightarrow \exists\ \mathcal{O}_X \overset{s}{\hookrightarrow} K_X \otimes V^*$

$\cong H^1(X,V)^*$. $\quad \mu(\mathcal{O}_X) = 0 \quad \mu(K_X\otimes V^*) = \dfrac{\deg}{\mathrm{rank}} < \dfrac{0}{n} < 0$

contradiction.

$$\zeta_{X,n}(s) := \underbrace{\sum_{m=0}^{2g-2} \sum_{[V]\in \mathcal{M}_{X,n}(mn)} \frac{q^{h^0(X,V)}}{\# \mathrm{Aut}(V)} (q^{-s})^{\chi(X,V)}}_{I(s)} - \underbrace{\sum_{m=0}^{\infty} \sum_{V\in \mathcal{M}_{X,n}(mn)} \frac{1}{\# \mathrm{Aut}\, V} \cdot t^{\chi(X,V)}}_{II(s)}$$

$$+ \sum_{m\geq 2g-1} \sum_{V\in \mathcal{M}_{X,n}(mn)} \frac{q^{h^0(X,V)} \overset{(ii)}{=} q^{\chi(X,V)}}{\# \mathrm{Aut}\, V} (q^{-s})^{\chi(X,V)} = III(s).$$

$$II(s) \overset{(i)}{=} \sum_{m=0}^{\infty} \sum_{V\in\mathcal{M}_{X,n}(mn)} \frac{1}{\#\mathrm{Aut}(V)} \cdot t^{mn-n(g-1)}$$

$$= \beta_{X,n}(0) \sum_{m=0}^{\infty} t^{(m-(g-1))n}$$

$$= \beta_{X,n}(0) \frac{T^{(1-g)}}{1-T}$$

$\mathcal{M}_{X,n}(0) \xrightarrow{\sim} \mathcal{M}_{X,n}(mn)$

$V \mapsto A^{\otimes m}\otimes V$

$\mathrm{Aut}(V) \cong \mathrm{Aut}(A^{\otimes m}\otimes V)$

$\Rightarrow$ $\boxed{\text{independent of } m}$

A: line bdl, $\deg(A) = 1$, A: $\mathbb{F}_q$-rational. Always exists!!!

Def: $\beta_{X,n}(0) = \sum_{[V]\in\mathcal{M}_{X,n}(0)} \dfrac{1}{\#\mathrm{Aut}(V)}$

$$III(s) = \beta_{X,n}(0) \sum_{m\geq 2g-1} \underbrace{(qt)^{mn-n(g-1)}}_{(QT)^{m-(g-1)}}$$

$2g-1$, $m-(g-1) \geq g$, $(QT)^{m-(g-1)}$

$$= \beta_{X,n}(0)\cdot \frac{\cancel{T}\cdot(QT)^g}{1-QT} = \beta_{X,n}(0)\cdot\frac{(QT)^g}{1-QT}$$

$$\frac{\left(\frac{1}{QT}\right)^{1-g}}{1-\frac{1}{QT}} = \frac{QT\,(QT)^{g-1}}{QT-1} = \frac{QT^g}{QT-1}$$

$\Rightarrow \boxed{III(1-s) = -II(s)}$.