Day 2. Special Uniformity of Zetas: $\hat{\zeta}_{X,n}(s) = \hat{\zeta}^{SL_n}_{X}(s)$ (2-1)

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I. Non-Abelian Zeta Function $\zeta_{X,n}(s)$

① Definition

$X/\mathbb{F}_q$, $q=p^r$: irreducible, reduced regular projective curve of genus g.

$\mathcal{M}_{X,n}(d)$: moduli stack of semi-stable bdl's of rank n & deg. d.

roughly: $\mathbb{F}_q$-rational bdls which are semistable.

Def. Rank n Non-Abelian Zeta Functions

$$\zeta_{X,n}(s) = \sum_{m\geq 0} \sum_{[V]\in \mathcal{M}_{X,n}(mn)} \frac{\#(H^0(X,V)\setminus\{0\})}{\#\mathrm{Aut}(V)} \cdot (q^{-s})^{\deg(V)} \qquad \mathrm{Re}(s)>1.$$

↖ Drinfeld

② Zeta Facts

Artin Zeta Function: $\zeta_X(s) = \sum_{D\geq 0} \frac{1}{ND^s} = \prod_{P\in X} \frac{1}{1-NP^{-s}}$

$\updownarrow$ valuation of F.

$\deg P = [k(P):k]$, $NP = q^{\deg P}$, $D = \sum n_i P_i \geq 0 \iff n_i \geq 0$ $\forall n_i = 0$. $\deg D = \sum n_i \deg(P_i)$, $ND = q^{\deg(D)}$

(1) $\zeta_X(s) = \sum_{d\geq 0} \sum_{[D_0]} \sum_{D\sim D_0} \frac{1}{ND^s}$, $[D_0] \in \mathrm{Pic}^d(X)$, $D = D_0 + \mathrm{div}(f)$

$$= \sum_{d\geq 0} \sum_{[D_0]\in \mathrm{Pic}^d(X)} \frac{|H^0(X,D_0)\setminus\{0\}|}{|\mathbb{F}_q^*|} (q^{-s})^{\deg(D)}$$

$\chi(X,D)$

$F \doteq \mathbb{F}_q(X)$

$f \in F^* \Rightarrow \mathrm{div}(f) = \sum_P \mathrm{ord}_P(f)\cdot P$

$\deg(f) = 0 \Rightarrow \deg(D) = \deg(D_0)$ if $D\sim D_0$

$$\left(= \sum_{d\geq 0} \sum_{L\in \mathrm{Pic}_d(X)} \frac{q^{h^0(X,L)}-1}{q-1} \cdot (q^{-s})^{\deg(L)} \right)$$

∵) $H^0(X,D_0) \ni f \mapsto \{D : D = D_0 + \mathrm{div}(f) \geq 0\}$

line bundle

$c\cdot f \mapsto \mathrm{div}(cf) + D_0 \iff f \mapsto \mathrm{div}(f) + D_0$

$\mathrm{Aut}(L) = \mathbb{F}_q^*$

Fact 0 $\boxed{\zeta_{X,1}(s) = \zeta_X(s)}$

Fact 1 Rationality: Rational Function of $t = q^{-s}$ in fact of $T = t^n$.

Fact 2 Functional Equation: $\hat{\zeta}_{X,n}(1-s) = \hat{\zeta}_{X,n}(s) := \zeta_{X,n}(s)\cdot (q^s)^{n(g-1)}$

Conj = Fact 3: $\zeta_{X,n}(s) = 0 \Rightarrow \mathrm{Re}(s) = 1/2$. Riemann Hypothesis. • $X = E$: elliptic curve (Weng-Zagier)