① Partitions: Parabolic Reduction

· G/X: reductive group scheme, $\mathcal{E}$: G-torsor $\Rightarrow \exists!$ canonical parabolic subgroup P s.t. $\mathcal{E}$ has its HN-type P.

$\Rightarrow$ · $\mathcal{M}^{total}_{X,G}(\nu_G')$: stack of G-torsors of slope ν_G', then $\mathcal{M}^{total}_{X,G}(\nu_G')$ admits a natural partition by the stack of $\mathcal{M}_{X,G,P}(\nu_P')$ ~~corresponding~~ to those having canonical type (P, ν_P') w/ $[\nu_P']_G = \nu_G'$, $[\nu_P']^G \in \mathfrak{a}_P^{G+}$.

In particular $\mathcal{M}_{X,G,G}(\nu_G') = \mathcal{M}_{X,G}(\nu_G') \leftarrow$ the moduli stack of semi-stable G-torsors of degree ν_G'.

· Set $\mathcal{M}^{total}_{X,G,Q}(\nu_Q') :=$ substack of $\mathcal{M}^{total}_{X,G}(\nu_G')$ induced from Q-torsors of slope ν_Q' ($[\nu_Q']_G = \nu_G'$).

Thm.
$$\mathcal{M}^{total}_{X,G,Q}(\nu_Q') = \bigcup_{P:P\subset Q} \bigcup_{\substack{\nu_P' \in X_*(A_P') \\ [\nu_P']_Q = \nu_Q' \\ [\nu_P']^Q \in \mathfrak{a}_P^{Q+}}} \mathcal{M}_{X,Q,P}(\nu_P').$$

Consequences: ①
$$\mathbb{1}_{\mathcal{M}^{total}_{X,G,Q}(\nu_Q')} = \sum_{P:P\subset Q} \sum_{\substack{\nu_P' \in X_*(A_P') \\ [\nu_P']_Q = \nu_Q'}} \tau_P^Q([\nu_P']^Q) \cdot \mathbb{1}_{\mathcal{M}_{X,Q,P}(\nu_P')}.$$

~~Adelically.~~
$$\cancel{\mathbb{1}_{\mathcal{M}^{total}_{X,G,Q}(\nu_Q')}(g) = \sum_{P:P\subset Q} \sum_{\substack{\nu_P' \in X_*(A_P') \\ [\nu_P']_Q = \nu_Q'}} \tau_P^Q([\nu_P']^Q) \quad \mathbb{1}_P^Q(\delta g)}$$
H_P'

②
$$\mathbb{1}_{\mathcal{M}_{X,G,Q}(\nu_Q')} = \sum_{P:P\subset Q} (-1)^{\dim \mathfrak{a}_P^Q} \sum_{\substack{\nu_P' \in X_*(A_P') \\ [\nu_P']_Q = \nu_Q'}} \hat{\tau}_P^Q([\nu_P']^Q) \cdot \mathbb{1}_{\mathcal{M}^{total}_{X,Q,P}(\nu_P')}.$$

__Example__ (Lafforgue) $G = SL_n$ ② is a result of Laurent. astérisque. 243.

$\boxed{\text{Adelic Version}}$

② Parabolic Subgroups & Their Levi factors.

Prop.
$$\mu(\mathcal{M}_{X,G,P}(\nu_P')) = \mathbb{L}^{2\langle \rho_P^G, \nu_P'\rangle + \dim(N_P)\cdot(g-1)} \cdot \mu(\mathcal{M}_{X,M_P}(\nu_P'))$$

Ex. $G = SL_n$. __Motivic Hall Algebra__. HN filtration $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$. $0 \to \mathcal{E}_1 \to \mathcal{E} \to E_1 \to 0$

· $\mathrm{Aut}\,\mathcal{E}_1 \times \mathrm{Aut}\,E_1$ acts on $\mathrm{Ext}^1(E_1, \mathcal{E}_1)$

· $\mathrm{Stab}_{\mathrm{Aut}\,\mathcal{E}_1 \times \mathrm{Aut}\,E_1} \mathcal{E}_x \simeq \mathrm{Aut}\,\mathcal{E} / (\mathrm{Id} + \mathrm{Hom}(E_1, \mathcal{E}_1)) \xleftarrow{!} $ canonical $\Rightarrow$ fixed $\mathcal{E}_1$ & E_1

$\Rightarrow \mathrm{Ext}^1(E_1, \mathcal{E}_1) = \bigcup_{[\mathcal{E}] \in \mathrm{Ext}^1(E_1, \mathcal{E}_1)} (\mathrm{Aut}\,\mathcal{E}_1 \times \mathrm{Aut}\,E_1) / (\mathrm{Aut}/(\mathrm{Id} + \mathrm{Hom}(E_1, \mathcal{E}_1)))$

$\Rightarrow \mathbb{L}^{\chi(X, E_1^\vee \otimes \mathcal{E}_1)} \int \frac{1}{\mu(\mathrm{Aut}(\mathcal{E}))} d\mu(\mathcal{E}) = \frac{1}{\mu(\mathrm{Aut}\,\mathcal{E}_1) \times \mu(\mathrm{Aut}\,E_1)}.$