Explicit Forms

Thm: $$\beta_{X,Q}(\nu_Q') = \sum_{\substack{P:\, P\subset Q \\ \text{standard parabolic}}} (-1)^{\dim \mathfrak{a}_P^Q} \cdot \hat{\beta}^{\text{total}}_{X,M_P}(s) \sum_{\substack{\lambda\in\Lambda_P^Q \\ [\lambda]_Q = \nu_Q'}} \prod_{\alpha\in\Delta_P^Q} \frac{\mathbb{L}^{2\langle \rho_P^Q, \alpha^\vee\rangle \{\varpi_\alpha^Q(\lambda)\}}}{\mathbb{L}^{2\langle \rho_P^Q, \alpha^\vee\rangle} - 1}$$

w/ $0 \le \{x\} = x - \lfloor x \rfloor < 1$

$0 < \langle x \rangle = 1 - \{x\} \le 1$. $\quad \Lambda_P^Q = X_*(A_P)/\sum_{\alpha\in\Delta_P^Q} \mathbb{Z}\alpha^\vee$.

& Similarly for $\hat{\beta}^{\text{total}}_{X,Q}(\nu_Q')$ in term of $\tilde{\beta}_{X,M_P}(\nu_P')$.

Laumon–Rapoport

Ex (H-N, D-R, Zagier)

$$\frac{\beta_{X,n}(d)}{\mathbb{L}^{\frac{n(n-1)}{2}(g-1)}} = \sum_{k\ge1} (-1)^{k-1} \sum_{\substack{n_1,\dots,n_k>0 \\ n_1+\dots+n_k=n}} \prod_{j=1}^{k-1} \frac{\mathbb{L}^{(n_j+n_{j+1})\{\frac{d}{n}(n_1+\dots+n_j)\}}}{\mathbb{L}^{n_j+n_{j+1}}-1} \cdot \prod_{i=1}^{k} \hat{\beta}_{X,n_i}(0)$$

$$\hat{\beta}_{X,n}(0) = \hat{\zeta}_X(1)\hat{\zeta}_X(2)\cdots\hat{\zeta}_X(n), \qquad \hat{\zeta}_X(1) = \frac{e(\mathrm{Pic}^0_X)}{\mathbb{L}-1}.$$

Thm. $k = \mathbb{F}_q$, $X = E$, elliptic curve

$$\sum_{n\ge1} \beta_{E,n}(0)\, q^{-ns} = \prod_{k\ge1} \zeta_E(s+k)$$

Multiplicative structure for β

$\Rightarrow$ RH for High rank

zeta functions

for elliptic curves.