$\omega : \Lambda^{top} \mathrm{Lie}\, G \rightsquigarrow \omega_x$ $\qquad \int_{\mathcal{G}(\mathcal{O}_p)} |\omega_p| = \mu(G(\mathbb{F}_p)) \cdot \mathbb{L}_p^{-\dim G}$ (4)

left & right invariant $\qquad$ when $\underline{p \text{ is nice}}$.

More generally, $\displaystyle\int_{\mathcal{G}(\mathcal{O}_x)} |\omega_x| := \mathbb{L}_x^{-\mathrm{ord}_x(\omega_x)} \cdot \underbrace{\mathbb{L}_x^{-\dim G}\, \mu(G_{k(x)})}$

where $\omega_x : T_e(\mathcal{G}(\mathcal{O}_x)) \to F_x$ $\qquad \mathrm{Im}(\omega_x) \overset{!!}{=} (\pi_x^{\mathrm{ord}_x(\omega_x)})$ ← fractional ideal

Def Motivic adelic measure

$$|\omega_{\mathbb{A}}| := \underbrace{\mathbb{L}^{(1-g)\dim G}}_{\text{compatible with Weil Restriction } R_{K/F}} \cdot \underbrace{\prod^{\omega}_{x \in X} |\omega_x|}_{\text{compatible w/ } \mathcal{G} = \mathbb{G}_a}$$

Conj. G: connected semisimple

$$\int_{G(F)\backslash G(\mathbb{A})} |\omega_{\mathbb{A}}| = 1.$$

Prop. $\mathbb{K} = \prod^{\omega}_{x \in X} \mathcal{G}(\mathcal{O}_x) \Rightarrow$

$$\mu(\mathbb{K}) = \int_{\mathbb{K}} |\omega_{\mathbb{A}}| = \mathbb{L}^{(1-g)\dim G} \cdot \frac{1}{\underbrace{\prod_{d \geq 1} Z_X(\mathbb{L}^{-d})^{\dim V_d}}_{= Z_X(M(G))}}$$

$M(G)$ ← motive associated to G (Gross Inv. Math)

Proof. (Motivic Euler product)

$$\int_{\mathbb{K}} |\omega_{\mathbb{A}}| = \underbrace{\left(\prod^{\omega}_{x \in X} \int_{\mathcal{G}(\mathcal{O}_x)} |\omega_x|\right)}_{\parallel} \cdot \mathbb{L}^{(1-g)\dim G}$$

$$\prod^{\omega}_{x \in X} \mathbb{L}_x^{-\mathrm{ord}_x(\omega_x)} \cdot \mathbb{L}_x^{-\dim G} \cdot \mu(G_{k(x)})$$

$$= \mathbb{L}^{-\deg(\omega)} \prod^{\omega}_{x \in X} \left(\prod_d (1 - \mathbb{L}_x^{-d})^{\dim V_d}\right)$$

$\boxed{\deg(\mathrm{Lie}\, \mathcal{G}) = 0}$ ← reductive gp