II, Motivic Measure: Adelic Space

②

$\mathbb{A} = \mathbb{A}_F = \prod^{\prime\,\omega}_{x\in X}(F_x, \mathcal{O}_x)$, $|\omega_x|$: motivic measure on F_x s.t.

$$\mu(\mathcal{O}_x) = \int_{\mathcal{O}_x} |\omega_x| = 1.$$

Basic Facts.

$$\begin{tikzcd} \mathcal{O}_x = \varprojlim_n \mathcal{O}_x/m_x^n \arrow[r] \arrow[d] \arrow[dr] & \mathcal{O}_x/m_x = k(x) \\ \mathcal{O}_x/m_x^n & \cdots\ \mathcal{O}_x/m_x^2 \end{tikzcd}$$

$$\underbrace{\{0\} \subset m_x^{n-1}/m_x^n \subset \cdots \subset m_x^1/m_x^n}_{} \subset \mathcal{O}_x/m_x^n$$

$$\underbrace{k(x),\quad k(x)\cdots k(x),\quad k(x)}_{n \text{ of them.}}$$

$$\mu(\mathcal{O}_x) = \mu(m_x^n)\cdot\mu(\mathcal{O}_x/m_x^n) = \mu(m_x^n)\cdot\mathbb{L}_x^n \quad \text{w/ } \mathbb{L}_x = \mathbb{L}^{\deg x}.$$

So $\boxed{\mu(m_x^n) = \mathbb{L}_x^{-n}}$

Keys.

Indeed, $\mathbb{A} = \varinjlim_{D_1}\varprojlim_{D_2 : D_2 \le D_1} \mathbb{A}(D_1)/\mathbb{A}(D_2)$ ← Ind-Pro topology {linearly compact locally

w/ $\mathbb{A}(D) = \{(a_x)\in\mathbb{A} : \mathrm{ord}_x(a_x) + n_x \ge 0 \ \ \forall x\}$. $D = \sum n_x x$

Key. $\mathbb{A}(D)$: linearly compact & open

$\mathbb{A}(D_1)/\mathbb{A}(D_2)$: finite dimensional very discrete & linearly compact.

Set $|\omega_{\mathbb{A}}| = \prod^{\omega}_{x\in X}\omega_x$ s.t. $\mu\left(\prod^{\omega}_{x\in X}\mathcal{O}_x\right) = \prod^{\omega}_{x\in X}\mu(\mathcal{O}_x) = \prod^{\omega}_{x\in X} 1 = 1.$

Thm: $\mu(\mathbb{A}/F) = \mathbb{L}^{g-1}$

∵) Nine Diagram

$$\begin{tikzcd} F\cap\mathbb{A}(D) \arrow[r, hook] \arrow[d] & \mathbb{A}(D) \arrow[r, two heads] \arrow[d] & \mathbb{A}(D)/F\cap\mathbb{A}(D) \arrow[d] \\ F \arrow[r, hook] \arrow[d] & \mathbb{A} \arrow[r, two heads] \arrow[d] & \mathbb{A}/F \arrow[d] \\ F/F\cap\mathbb{A}(D) \arrow[r, hook] & \mathbb{A}/\mathbb{A}(D) \arrow[r, two heads] & \mathbb{A}/\mathbb{A}(D)+F \end{tikzcd}$$

Lem. $D \ge 0$

(i) $\mathbb{A}(D)/\mathbb{A}(0) \simeq \mathbb{L}^{\deg D} \simeq \prod_{x\in X}\mathbb{L}_x^{n_x}$

(ii) $\mathbb{A}(0) = \prod^{\omega}_{x\in X}\mathcal{O}_x$

(iii) $H^0(X, D) = F\cap\mathbb{A}(D)$

(iv) $H^1(X, D) = \mathbb{A}/\mathbb{A}(D)+F$.