Motivic Euler Product & Its Applications

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(I) Motivic Zetas & Motivic Euler Products

k: field X/k: irreducible, reduced reg. proj curve of genus g

$F = k(X)$, $x \in X$: closed point $\Rightarrow$ F_x, $\mathcal{O}_x$, m_x, $k(x)$, $\deg(x) = [k(x):k]$.

$\mathcal{M}_{X,n}(d)$: moduli stack of s-stable bundles of rank n & deg d.

<u>Def (1)</u>
$$Z_{X,n}(u) = \sum_{m\geq 0} \int^{\#}_{[V]\in \mathcal{M}_{X,n}(mn)} \frac{\mu(H^0(X,V)\setminus\{0\})}{\mu(\mathrm{Aut}(V))}\, d\mu \cdot u^{\deg(V)}$$

$\in K_0(ST_{/k})[[u]]$ Motivic Grothendieck K-Ring

$\mathbb{L} = \mu(\mathbb{A}^1_k)$

<u>e.g.</u> $\mu(\mathbb{P}^1_k) = \mathbb{L}+1$

$\mu(\mathbb{G}_{m,k}) = \mathbb{L}-1$

$\mu(GL_n(k)) = (\mathbb{L}^n-1)(\mathbb{L}^n-\mathbb{L})\cdots(\mathbb{L}^n-\mathbb{L}^{n-1})$.

$$\begin{cases} \mu(X) = \mu(X-Z) + \mu(Z) & \forall Z \hookrightarrow X: \text{closed} \\ \mu(X\times Y) = \mu(X)\cdot\mu(Y) \\ \mu(X/G) = \mu(X)/\mu(G) \end{cases}$$

In fact, $K_0(ST_{/k}) = \widehat{K_0(Var_{/k})}^{\mathbb{L}}$ $\mathbb{L}$-adic completion.

<u>Fact</u>. $Z_{X,1}(u) = Z_X(u) := \sum_{m\geq 0} \mu(\mathrm{Sym}^{(m)}X)\cdot u^m$ Motivic Zeta Function.

<u>Def (2)</u> Motivic Euler Product $\forall X/k$ variety.

$$\prod_{x\in X}^{w} \frac{1}{1-u_x} := \sum_{m\geq 0} \mu(\mathrm{Sym}^{(m)}X)\cdot u^m \quad \text{w/ } u_x = u^{\deg(x)}$$

<u>Ex</u> (i) $k = \mathbb{F}_q \Rightarrow \prod^{w}_{x\in X}$ ordinary Euler product being countable.

(ii) $k = \bar{k} = \mathbb{C}$ e.g. $\rightsquigarrow$ New object.

<u>Basic Properties</u>

(i) $\prod^{w}_{x\in X} f(x) = \prod'_{x\in X} f(x)$ if $\forall' x$, $f(x)=1$ (ii) $\prod^{w}_{x\in X} \frac{1}{1-u_x} = \prod^{w}_{x\in Z} \frac{1}{1-u_x} \cdot \prod^{w}_{x\in U} \frac{1}{1-u_x}$