Arithmetic Cohomology Groups

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Abstract

We first introduce global arithmetic cohomology groups for quasi-coherent sheaves on arithmetic varieties, adopting an adelic approach. Then, we establish fundamental properties, such as topological duality and inductive long exact sequences, for these cohomology groups on arithmetic surfaces. Finally, we expose basic structures for ind-pro topologies on adelic spaces of arithmetic surfaces. In particular, we show that these adelic spaces are topologically self-dual.

Contents

Introduction

In the study of arithmetic varieties, cohomology theory has been developed along with the line of establishing an intrinsic relation between arithmetic Euler characteristics and arithmetic intersections. For examples, for an arithmetic curve $\mathrm{Spec}\,\mathcal{O}_F$ associated to the integer ring $\mathcal{O}_F$ of a number field F with discriminant Δ_F and a metrized vector sheaf $\overline{E}$ on it, we have the Arakelov-Riemann-Roch formula

$$\chi_{\mathrm{ar}}(F,\overline{E}) = \deg_{\mathrm{ar}}(\overline{E}) - \frac{\mathrm{rank}E}{2}\log|\Delta_F|;$$

And, for a regular arithmetic surface $\pi : X \to \mathrm{Spec}\,\mathcal{O}_F$ and a metrized line sheaf $\overline{\mathcal{L}}$ on it, if we equip with X_∞ a Kähler metric, and line sheaves $\lambda(\mathcal{L})$ and $\lambda(\mathcal{O}_X)$ with the Quillen metrics, namely, equip determinants of relative cohomology groups with determinants of L^2-metrics modified by analytic torsions, then we have the Faltings-Deligne-Riemann-Roch isometry

$$\overline{\lambda(\mathcal{L})}^{\otimes 2} \otimes \overline{\lambda(\mathcal{O}_X)}^{\otimes -2} \simeq \langle \overline{\mathcal{L}}, (\overline{\mathcal{L}} \otimes \overline{K}_\pi^{\otimes -1})\rangle;$$

More generally, for higher dimensional arithmetic varieties, we have the works of (Bismut-)Gillet-Soulé.

In this paper, we start to develop a genuine cohomology theory for arithmetic varieties, as a continuation of the works of Parshin ([P1,2]), Beilinson ([B]), Osipov-Parshin ([OP]), and our own study ([W]). Our aims here are to construct arithmetic cohomology groups H^i_{ar} for quasi-coherent sheaves on arithmetic varieties, and to establish topological dualities among these cohomology groups for arithmetic surfaces.

The approach we take in this paper is an adelic one. Here, we use three main ideas form the literatures. Namely, the first one of adelic complexes initiated in the classical works [P1,2] and [B], (see also [H]), which is recalled in §1.1 and used in §1.2 systematically; the second one of ind-pro structures over adelic spaces from [OP], which is recalled in §1.2.1 and motivates our general constructions in §1.2.2; and the final one on the uniformity structure between finite and infinite adeles from [W], which is recalled in §1.2.4 and plays an essential role in §1.2.3

when we construct our adelic spaces. In particular, we are able to introduce arithmetic adelic complexes $(\mathbb{A}^*_{\mathrm{ar}}(X,\mathcal{F}),d^*)$ for quasi-coherent sheaves $\mathcal{F}$ over arithmetic varieties X, and hence are able to define their associated arithmetic cohomology groups $H^i_{\mathrm{ar}}(X,\mathcal{F}) := H^i(\mathbb{A}^*_{\mathrm{ar}}(X,\mathcal{F}),d^*)$. Consequently, we have the following

Theorem I. *Let X be an arithmetic variety and $\mathcal{F}$ be a quasi-coherent sheaf on X, then there exist a natural arithmetic adelic complex $(\mathbb{A}^*_{\mathrm{ar}}(X,\mathcal{F}),d^*)$ and hence arithmetic cohomology groups $H^i_{\mathrm{ar}}(X,\mathcal{F}) := H^i(\mathbb{A}^*_{\mathrm{ar}}(X,\mathcal{F}),d^*)$. In particular, $H^i_{\mathrm{ar}}(X,\mathcal{F}) = 0$ unless $i = 0,1,\dots,\dim X_{\mathrm{ar}}$.*

To understand this general cohomology theory in down-to-earth terms, in section two, we develop a much more refined cohomology theory for Weil divisors D over arithmetic surfaces X. This, in addition, is based on a basic theory for canonical ind-pro topologies over arithmetic adelic spaces. Recall that, by definition,

$$\mathbb{A}^{\mathrm{ar}}_X := \mathbb{A}^{\mathrm{ar}}_X(\mathcal{O}_X) \simeq \lim_{\longrightarrow D} \lim_{\substack{\longleftarrow E \\ E\leq D}} \mathbb{A}^{\mathrm{ar}}_{X,12}(D)\big/\mathbb{A}^{\mathrm{ar}}_{X,12}(E).$$

Here $\mathbb{A}^{\mathrm{ar}}_{X,12}(D)$ is one of the level two subspaces of $\mathbb{A}^{\mathrm{ar}}_X$ introduced in §2.3.1. Moreover, for divisors $E \leq D$, $\mathbb{A}^{\mathrm{ar}}_{X,12}(D)\big/\mathbb{A}^{\mathrm{ar}}_{X,12}(E)$ are locally compact. Thus, using first projective then inductive limits, we obtain a canonical ind-pro topology on $\mathbb{A}^{\mathrm{ar}}_X$. In particular, we have the following natural generalization of topological theory for one dimensional adeles (see e.g., [Iw], [T]) to dimension two.

Theorem II. *Let X be an arithmetic surface. With respect to the canonical ind-pro topology on $\mathbb{A}^{\mathrm{ar}}_X$, we have*

(1) $\mathbb{A}^{\mathrm{ar}}_X$ is a Hausdorff, complete, and compact oriented topological group;

(2) $\mathbb{A}^{\mathrm{ar}}_X$ is self-dual. That is, as topological groups,

$$\widehat{\mathbb{A}^{\mathrm{ar}}_X} \simeq \mathbb{A}^{\mathrm{ar}}_X.$$

With these basic topological structures exposed, next we apply them to our cohomology groups. For this, we first recall an arithmetic residue theory in §2.1, by adopting a very precise approach of Morrow [M1,2], a special case of a general theory on residues of Grothendieck (see e.g., [L], [B] and [Y]). Then, we introduce a global pairing $\langle\cdot,\cdot\rangle_\omega$ in §2.2 on the arithmetic adelic space $\mathbb{A}^{\mathrm{ar}}_X$, and prove the following

Proposition A. *Let X be an arithmetic surface and ω be a non-zero rational differential on X. Then, the natural residue pairing $\langle\cdot,\cdot\rangle_\omega : \mathbb{A}^{\mathrm{ar}}_X \times \mathbb{A}^{\mathrm{ar}}_X \to \mathbb{S}^1$ is non-degenerate.*

Moreover, we construct the so-called level two adelic subspaces $\mathbb{A}^{\mathrm{ar}}_{X,01}, \mathbb{A}^{\mathrm{ar}}_{X,02}$ and $\mathbb{A}^{\mathrm{ar}}_{X,12}(D)$ of $\mathbb{A}^{\mathrm{ar}}_X$ in §2.3.1. Accordingly, we calculate their perpendicular subspaces with respect to our global residue pairing.

Proposition B. *Let X be an arithmetic surface, D be a Weil divisor and ω be a non-zero rational differential on X. Denote by (ω) the canonical divisor on X associated to ω. Then we have*

(1) Level two subspaces $\mathbb{A}^{\mathrm{ar}}_{X,01}, \mathbb{A}^{\mathrm{ar}}_{X,02}$ and $\mathbb{A}^{\mathrm{ar}}_{X,12}(D)$ are closed in $\mathbb{A}^{\mathrm{ar}}_X$;

(2) With respect to the residue pairing $\langle\cdot,\cdot\rangle_\omega$,

$$\big(A^{\mathrm{ar}}_{X,01}\big)^\perp = A^{\mathrm{ar}}_{X,01}, \quad \big(A^{\mathrm{ar}}_{X,02}\big)^\perp = A^{\mathrm{ar}}_{X,02}, \quad \text{and} \quad \big(A^{\mathrm{ar}}_{X,12}(D)\big)^\perp = A^{\mathrm{ar}}_{X,12}((\omega)-D).$$

Our lengthy proof for (2) is based on the residue formulas for horizontal and vertical curves on arithmetic surfaces established in [M2]. Moreover, as one can find from the proof of this theorem in §2.3.2, all the level two adelic subspaces $\mathbb{A}^{\rm ar}_{X,01}, \mathbb{A}^{\rm ar}_{X,02}$ and $\mathbb{A}^{\rm ar}_{X,12}(D)$ are characterized by these perpendicular properties as well. As for (1), our proof in §3.1.3 uses a topological notion of completeness in an essential way.

With the help of these level two subspaces, now we are ready to write down the adelic complex of §1.2.3 and hence its cohomology groups $H^i_{\rm ar}(X, \mathcal{O}_X(D))$ associated to the line bundle $\mathcal{O}_X(D)$ on an arithmetic surface X very explicitly. Indeed, according to §1.2.3, or more directly, following [P], we arrive at the following central

Definition. *Let X be an arithmetic surface and D be a Weil divisor on X. We define arithmetic cohomology groups $H^i_{\rm ar}(X, \mathcal{O}_X(D))$ for the line bundle $\mathcal{O}_X(D)$ on X, $i = 0, 1, 2$, by*

$$H^0_{\rm ar}(X, \mathcal{O}_X(D)) := \mathbb{A}^{\rm ar}_{X,01} \cap \mathbb{A}^{\rm ar}_{X,02} \cap \mathbb{A}^{\rm ar}_{X,12}(D);$$

$$H^1_{\rm ar}(X, \mathcal{O}_X(D))$$

$$:= \Big(\big(\mathbb{A}^{\rm ar}_{X,01} + \mathbb{A}^{\rm ar}_{X,02}\big) \cap \mathbb{A}^{\rm ar}_{X,12}(D) \Big) \Big/ \Big(\mathbb{A}^{\rm ar}_{X,01} \cap \mathbb{A}^{\rm ar}_{X,12}(D) + \mathbb{A}^{\rm ar}_{X,02} \cap \mathbb{A}^{\rm ar}_{X,12}(D) \Big);$$

$$H^2_{\rm ar}(X, \mathcal{O}_X(D)) := \mathbb{A}^{\rm ar}_{X,012} \Big/ \Big(\mathbb{A}^{\rm ar}_{X,01} + \mathbb{A}^{\rm ar}_{X,02} + \mathbb{A}^{\rm ar}_{X,12}(D) \Big).$$

Similar to usual cohomology theory, these cohomology groups admit a natural inductive structure. For details, please refer to Propositions 17, 18 in §2.4.2. Moreover, induced from the canonical ind-pro topology on $\mathbb{A}^{\rm ar}_X$, we obtain natural topological structures for our cohomology groups, since, from Proposition B(1) above, the subspaces $\mathbb{A}^{\rm ar}_{X,01}, \mathbb{A}^{\rm ar}_{X,02}$ and $\mathbb{A}^{\rm ar}_{X,12}(D)$ are all closed. Consequently, as one of the main results of this paper, with the use of Theorem II above, in §3.2.4, we are able to establish the following

Theorem III. *Let X be an arithmetic surface with a canonical divisor K_X and D be a Weil divisor on X. Then, as topological groups,*

$$H^i_{\rm ar}(\widehat{X, \mathcal{O}_X}(D)) \simeq H^{2-i}_{\rm ar}(X, \mathcal{O}_X(K_X - D)) \qquad i = 0, 1, 2.$$

Our theory is natural and proves to be very useful. For example, as recalled in §1.2.4, in [W], based on Tate's thesis ([T]), for a metrized bundle $\overline{E}$ on an arithmetic curve $\operatorname{Spec} \mathcal{O}_F$, we are able to prove a refined arithmetic duality:

$$h^1_{\rm ar}\big(X, \overline{E}\big) = h^0_{\rm ar}\big(X, \overline{K_X} \otimes \overline{E}^\vee\big),$$

and establish 'the' arithmetic Riemann-Roch theorem:

$$h^0_{\rm ar}\big(X, \overline{E}\big) - h^1_{\rm ar}\big(X, \overline{E}\big) = \deg_{\rm ar}(\overline{E}) - \frac{{\rm rank} E}{2} \log |\Delta_F|,$$

(where h^i denotes the arithmetic count of $H^i_{\rm ar}$,) and obtain an effective version of ampleness and vanishing theorem. All this plays an essential role in our studies of non-abelian zeta functions for number fields.

1 Arithmetic Adelic Complexes

1.1 Parshin-Beilinson's Theory

For later use, we here recall some basic constructions of adelic cohomology theory for Noetherian schemes of Parshin-Beilinson ([P1,2], [B], see also [H]).

1.1.1 Local fields for reduced flags

Let F be a number field with $\mathcal{O}_F$ the ring of integers, and $\pi : X \to \mathrm{Spec}\,\mathcal{O}_F$ be an integral arithmetic variety. By a flag $\delta = (p_0, p_1, \ldots, p_n)$ on X, we mean a chain of integral subschemes p_i satisfying $p_{i+1} \in \overline{\{p_i\}} =: X_i$; and we call δ reduced if $\dim p_i = n - i$ for each i. For a reduced δ, with respect to each affine open neighborhood $U = \mathrm{Spec}\,B$ of the closed point p_n, we obtain a chain, denoted also by δ with an abuse of notation, of prime divisors on U. Consequently, through processes of localizations and completions, we can associate to δ a ring

$$K_\delta := C_{p_0} S_{p_0}^{-1} \ldots C_{p_n} S_{p_n}^{-1} B.$$

Here, as usual, for a ring R, an R-module M and a prime ideal p of R, we write $S_p^{-1} M$ for the localization of M at $S_p = R \backslash p$, and $C_p M = \lim_{\leftarrow n \in \mathbb{N}} M/p^n M$ its p-adic completion.

The ring K_δ is independent of the choices of B. Indeed, following [P2], we can introduce inductively schemes X'_{i,α_i} as in the following diagram

$$\begin{array}{ccccccc} X_0 & \supset & X_1 & \supset & X_2 & \supset & \cdots \\ \uparrow & & \uparrow & & & & \\ X'_0 & \supset & X_{1,\alpha_1} & \supset & \uparrow & & \\ & & \uparrow & & & & \\ & & X'_{1,\alpha_1} & \supset & X_{2,\alpha_2} & & \\ & & & & \uparrow & & \\ & & & & \vdots & & \end{array}$$

where X' denotes the normalization of a scheme X, and X_{i,α_i} denotes an integral subscheme in $X'_{i-1,\alpha_{i-1}}$ which dominates X_i. In particular,
(i) X_{1,α_1}, being an integral subvariety of the normal scheme X'_0, defines a discrete valuation of the field of rational functions on X_0, whose residue field coincides with the field of rational functions on the normal scheme X'_{1,α_1}.
(ii) More generally, for a fixed i, $1 \leq i \leq n$, X_{i,α_i}, being an integral subvariety of the normal scheme $X'_{i-1,\alpha_{i-1}}$, defines a discrete valuation of the field of rational functions on $X'_{i-1,\alpha_{i-1}}$, whose residue field coincides with the field of rational functions on the normal scheme X'_{i,α_i}.
Accordingly, for each collection $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of indices, the chain of field of rational functions $K_0, K_{1,\alpha_1}, \ldots, K_{n,\alpha_n}$ defines an n-dimensional local field $K_{(\alpha_1,\ldots,\alpha_n)}$ and hence an Artin ring

$$\mathbb{K}_\delta := \oplus_{(\alpha_1,\ldots,\alpha_n)\in\Lambda_\delta} K_{(\alpha_1,\ldots,\alpha_n)}.$$

Theorem 1. *([P1,2], [Y]) Let $\delta = (p_0, p_1, \ldots, p_n)$ be a reduced flag on X, and $K_{(\alpha_1,\ldots,\alpha_n)}$ be the n-dimensional local field associated to the collection of indices $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ above. Then, we have*

(1) The n-dimensional local field $K_{(\alpha_1,\ldots,\alpha_n)}$ is, up to finite extension, isomorphic to

$$\text{either} \quad F'_v((t_{n-1}))\cdots((t_1)), \quad \text{or} \quad F'_v\{\{t_n\}\}\cdots\{t_{m+2}\}\}((t_m))\ldots((t_1))$$

where F'_v denotes a certain finite extension of some v-adic non-Archimedean local field F_v;
(2) The ring K_δ is isomorphic to $\mathbb{K}_\delta$. In particular, it is independent of the choices of U.

For example, if X is an arithmetic surface, and p_1 is a vertical curve, then, up to finite extension, $K_\delta = F_{\pi(p_2)}\{\{u\}\}$,[1] where u denotes a local parameter of the curve p_1 at the point p_2, and $F_{\pi(p_2)}$ denotes the $\pi(p_2)$-adic number field associated to the closed point $\pi(p_2)$ on $\operatorname{Spec}\mathcal{O}_F$; on the other hand, if p_1 is a horizontal curve, then $K_\delta = L((t))$, where t is a local parameter of p_1 at p_2, and L/F is a finite field extension. Indeed, p_1 corresponds to an algebraic point on the generic fiber X_F of π, and L is simply the corresponding defining field.

1.1.2 Adelic cohomology theory

Let X be a Noetherian scheme, and let $P(X)$ be the set of (integral) points of X (in the scheme theoretic sense). For $p,\, q \in P(X)$, if $q \in \overline{\{p\}}$, we write $p \geq q$. Let $S(X)$ be the simplicial set induced by $(P(X), \geq)$, i.e., the set of m-simplices of $S(X)$ is defined by

$$S(X)_m := \big\{(p_0,\ldots,p_m) \mid p_i \in P(X),\ p_i \geq p_{i+1}\big\},$$

the natural boundary maps δ_i^n are defined by deleting the i-th point, and the degeneracy maps σ_i^n are defined by duplicating the i-th point:

$$\delta_i^m : S(X)_m \to S(X)_{m-1},\ \ (p_0,\ldots,p_i,\ldots,p_m) \mapsto (p_0,\ldots,\check{p}_i,\ldots,p_m),$$

$$\sigma_i^m : S(X)_m \to S(X)_{m+1},\ \ (p_0,\ldots,p_i,\ldots,p_m) \mapsto (p_0,\ldots,p_i,p_i,\ldots,p_m).$$

Denote also by $S(X)_m^{\mathrm{red}}$ the subset of $S(X)_m$ consisting of all non-degenerate simplifies, i.e.,

$$S(X)_m^{\mathrm{red}} = \big\{(p_0,\ldots,p_m) \in S(X)_m \;\big|\; \dim p_i \neq \dim p_j \ \ \forall i \neq j\;\big\}.$$

For $p \in P(X)$ and M an $\mathcal{O}_p$-module, set $[M]_p := (i_p)_* M$, where $i_p : \operatorname{Spec}(\mathcal{O}_p) \hookrightarrow X$ denotes the natural induced morphism. Moreover, for $K \subset S(X)_m$ and a point $p \in P(X)$, introduce ${}_pK \subset S(X)_{m-1}$ by

$${}_pK := \big\{(p_1,\ldots,p_m) \in S(X)_{m-1} \mid (p,p_1,\ldots,p_m) \in K\big\}.$$

Then, we have the following

Proposition 2. *([P1,2], [B], see also [H, Prop 2.1.1]) There exists a unique system of functors $\{\mathbb{A}(K,*)\}_{K\subset S(X)}$ from the category of quasi-coherent sheaves on X to the category of abelian groups, such that*
(i) $\mathbb{A}(K,\cdot)$ commutes with direct limits.

[1]Definition of $F_{\pi(p_2)}\{\{u\}\}$ will be recalled in §2.1.1.

(ii) For a coherent sheaf $\mathcal{F}$ on X,

$$\mathbb{A}(K,\mathcal{F}) = \begin{cases} \prod_{p\in K} \lim_{\longleftarrow l} \mathcal{F}_p/\mathfrak{m}_p^l\mathcal{F}_p, & m=0, \\ \prod_{p\in P(X)} \lim_{\longleftarrow l} \mathbb{A}\big({}_pK, [\mathcal{F}_p/\mathfrak{m}_p^l\mathcal{F}_p]_p\big), & m>0. \end{cases}$$

Here $\mathfrak{m}_p$ denotes the prime ideal associated to p.

Consequently, for any quasi-coherent sheaf $\mathcal{F}$ on X, there exist well-defined adelic spaces

$$\mathbb{A}_X^m(\mathcal{F}) := \mathbb{A}\big(S(X)_m^{\mathrm{red}}, \mathcal{F}\big).$$

Clearly, if we introduce $K_{i_0,\ldots,i_m} = \big\{(p_0,\ldots,p_m) \in S(X)_m \,|\, \mathrm{codim}\big(\overline{\{p_r\}}\big) = i_r \ \forall\, 0 \leq r \leq m\big\}$, and define $\mathbb{A}_{X;i_0,\ldots,i_m}(\mathcal{F}) := \mathbb{A}_X\big(K_{i_0,\ldots,i_m}, \mathcal{F}\big)$, then

$$\mathbb{A}_X^m(\mathcal{F}) = \bigoplus_{0\leq i_0<\cdots<i_m\leq \dim X} \mathbb{A}_{X;i_0,\ldots,i_m}(\mathcal{F}).$$

Moreover, since $\mathbb{A}(K,\mathcal{F}) \subset \prod_{(p_0,\ldots,p_m)\in K} \mathbb{A}\big((p_0,\ldots,p_m),\mathcal{F}\big)$, we sometimes write an element f of $\mathbb{A}(K,\mathcal{F})$ as $f = (f_{p_0,\ldots,p_m})$ or $f = (f_{X_0,\ldots,X_m})$, where $X_i = \overline{\{p_i\}}$ and $f_{p_0,\ldots,p_m} = f_{X_0,\ldots,X_m} \in \mathbb{A}\big((p_0,\ldots,p_m),\mathcal{F}\big)$.

To get an adelic complex associated to X, we next introduce boundary maps $d^m : \mathbb{A}_X^{m-1}(\mathcal{F}) \to \mathbb{A}_X^m(\mathcal{F})$ as in [H, Def 2.2.2]. For $K \subset S(X)_m$ and $L \subset S(X)_{m-1}$ such that $\delta_i^m K \subset L$ for a certain i, we define a boundary map

$$d_i^m(K,L,\mathcal{F}) : \ \mathbb{A}(L,\mathcal{F}) \longrightarrow \mathbb{A}(K,\mathcal{F})$$

as follows.

(a) For coherent sheaves $\mathcal{F}$,

(i) When $i=0$, for $p \in P(X)$, induced from the morphism $\mathcal{F} \to [\mathcal{F}_p/\mathfrak{m}_p^l\mathcal{F}_p]_p$ and the inclusion ${}_pK \subset L$, we have the morphisms $\mathbb{A}(L,\mathcal{F}) \to \mathbb{A}\big(L, [\mathcal{F}_p/\mathfrak{m}_p^l\mathcal{F}_p]_p\big)$ and $\mathbb{A}\big(L, [\mathcal{F}_p/\mathfrak{m}_p^l\mathcal{F}_p]_p\big) \to \mathbb{A}\big({}_pK, [\mathcal{F}_p/\mathfrak{m}_p^l\mathcal{F}_p]_p\big)$. Their compositions form a projective system $\varphi_p^l : \mathbb{A}(L,\mathcal{F}) \to \mathbb{A}\big({}_pK, [\mathcal{F}_p/\mathfrak{m}_p^l\mathcal{F}_p]_p\big)$. Accordingly, we set $d_0^m(K,L,\mathcal{F}) := \prod_{p\in P(X)} \lim_{\longleftarrow l} \varphi_p^l$;

(ii) When $i=m=1$, we obtain a projective system induced from the standard morphisms $\pi_p^l : \Gamma\big(X, [\mathcal{F}_p/\mathfrak{m}_p^l\mathcal{F}_p]_p\big) \to \mathbb{A}\big({}_pK, [\mathcal{F}_p/\mathfrak{m}_p^l\mathcal{F}_p]_p\big)$. Accordingly, we set $d_1^1(K,L,\mathcal{F}) := \prod_{p\in P(X)} \lim_{\longleftarrow l} \pi_p^l$;

(iii) When $i>0, m>0$, we use an induction on (i,m). That is to say, we set $d_i^m(K,L,\mathcal{F}) := \prod_{p\in P(X)} \lim_{\longleftarrow l} d_{i-1}^{m-1}\big({}_pK, {}_pL, [\mathcal{F}_p/\mathfrak{m}_p^l\mathcal{F}_p]_p\big)$.

(b) For quasi-coherent sheaves $\mathcal{F}$, first we write $\mathcal{F}$ as an inductive limit of coherent sheaves, then we use (a) to get boundary maps for the later, finally we use the fact that in the definition of (a), all constructions commute with inductive limits. One checks (see e.g. [H]) that the resulting boundary map is well-defined.

With this, set

$$d^m := \sum_{i=0}^m (-1)^i\, d_i^m\big(S(X)_m^{\mathrm{red}}, S(X)_{m-1}^{\mathrm{red}}; \mathcal{F}\big).$$

Then we have the following

Theorem 3. *([P1,2], [B], see also [H, Thm 4.2.3]) Let X be a Noetherian scheme. Then, for any quasi-coherent sheaf $\mathcal{F}$ over X, we have*
(1) $\big(\mathbb{A}_X^*(\mathcal{F}), d^*\big)$ forms a cohomological complex of abelian groups;
(2) Cohomology groups of the complex $\big(\mathbb{A}_X^*(\mathcal{F}), d^*\big)$ coincide with Grothendieck's sheaf theoretic cohomology groups $H^i\big(X, \mathcal{F}\big)$. That is to say, we have, for all i,

$$H^i\big(\mathbb{A}_X^*(\mathcal{F}),\, d^*\big) \simeq H^i\big(X, \mathcal{F}\big).$$

1.1.3 An example

Let X be an integral regular projective curve defined over a field k. Denote its generic point by η and its field of rational functions by $k(X)$. For a divisor D on X, let $\mathcal{O}_X(D)$ be the associated invertible sheaf. Then, from definition, the associated adelic spaces can be calculated as follows:

$$\begin{aligned}
\mathbb{A}_{X;0}(\mathcal{O}_X(D)) &= \mathbb{A}\big(\{\eta\}, \mathcal{O}_X(D)\big)\\
&= \lim_{\longleftarrow_l} \mathcal{O}_X(D)_\eta / \mathfrak{m}_\eta^l \mathcal{O}_X(D)_\eta = \lim_{\longleftarrow_l} k(X)/\{0\} = k(X),\\
\mathbb{A}_{X;1}(\mathcal{O}_X(D)) &= \mathbb{A}_X\big(\{p\} \,|\, p \in X :\ \text{closed point}\}, \mathcal{O}_X(D)\big)\\
&= \prod_{p\in X} \lim_{\longleftarrow_l} \mathcal{O}_X(D)_p / \mathfrak{m}_p^l \mathcal{O}_X(D)_p = \prod_{p\in X} \lim_{\longleftarrow_l} \mathfrak{m}_p^{-\mathrm{ord}_p(D)} / \mathfrak{m}_p^{-\mathrm{ord}_p(D)+l}\\
&= \prod_{p\in X} \mathfrak{m}_p^{-\mathrm{ord}_p(D)} = \Big\{(a_p) \in \prod_{p\in X} k(X)_p \,\Big|\, \mathrm{ord}_p(a_p) + \mathrm{ord}_p(D) \geq 0\Big\},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{A}_{X;01}(\mathcal{O}_X(D)) &= \mathbb{A}_X\big(\{\eta, p \,|\, p \in X :\ \text{closed point}\}, \mathcal{O}_X(D)\big)\\
&= \lim_{\longleftarrow l} \mathbb{A}\big(\{p\} \,|\, p \in X :\ \text{closed point}\}, [\mathcal{O}_X(D)_\eta / \mathfrak{m}_\eta^l \mathcal{O}_X(D)_\eta]_\eta\big)\\
&= \mathbb{A}\big(\{p\} \,|\, p \in X :\ \text{closed point}\}, [k(X)]_\eta\big)\\
&= \mathbb{A}\big(\{p\} \,|\, p \in X :\ \text{closed point}\},\ \lim_{\longrightarrow E} \mathcal{O}_X(E)\big)\\
&= \lim_{\longrightarrow E} \mathbb{A}_{X;1}(\mathcal{O}_X(E)) = \bigcup_E \mathbb{A}_{X;1}(\mathcal{O}_X(E))\\
&= \Big\{(a_p) \in \prod_{p\in X} k(X)_p \,\Big|\, a_p \in \mathcal{O}_p\ \forall' p\Big\}.
\end{aligned}$$

Remark. To calculate $\mathbb{A}_{X;01}(\mathcal{O}_X(D))$, when dealing with the constant sheaf $[k(X)]_\eta$, we cannot use Proposition 2(ii) directly, since $[k(X)]_\eta$ is not coherent. Instead, above, we first expressed it as an inductive limit of coherent sheaves $\mathcal{O}_X(E)$ associated to divisors E, then get the result from the inductive limit of adelic spaces for $\mathcal{O}_X(E)$'s. Indeed, if we had used Proposition 2 (ii) directly, then we would have obtained simply $\mathbb{A}\big(\{p\} \,|\, p \in X :\ \text{closed point}\}, [k(X)]_\eta\big) = \{0\}$, a wrong claim.

Clearly, $\mathbb{A}_{X;01}(\mathcal{O}_X(D))$ is independent of D. We will write it as $\mathbb{A}_{X;01}$, or simply $\mathbb{A}_X$. Consequently, the associated adelic complex

$$0 \longrightarrow \mathbb{A}_{X;0}(\mathcal{O}_X(D)) \oplus \mathbb{A}_{X;1}(\mathcal{O}_X(D)) \xrightarrow{d^1} \mathbb{A}_{X;01}(\mathcal{O}_X(D)) \longrightarrow 0$$

is given by

$$0 \longrightarrow k(X) \oplus \mathbb{A}_{X;1}(\mathcal{O}_X(D)) \xrightarrow{d^1} \mathbb{A}_X \longrightarrow 0$$

where $d^1 : (a_0, a_1) \mapsto a_1 - a_0$. Therefore,

$$H^0\big(\mathbb{A}_X(\mathcal{O}_X(D))\big) = k(X) \cap \mathbb{A}_{X;1}(\mathcal{O}_X(D)),$$
$$H^1\big(\mathbb{A}_X(\mathcal{O}_X(D))\big) = \mathbb{A}_X\big/\big(k(X) + \mathbb{A}_{X;1}(\mathcal{O}_X(D))\big).$$

Note that $\mathbb{A}_X(\mathcal{O}_X(D))$ is simply $\mathbb{A}_X(D)$ of [S, Ch. 2], or better, [Iw, §4]. We have proved the following

Proposition 4. *(See e.g., [S, Ch. 2], [Iw, §4]) For a divisor D over an integral regular projective curve defined over a field k, we have*

$$H^0\big(\mathbb{A}_X(\mathcal{O}_X(D))\big) = H^0\big(X, \mathcal{O}_X(D)\big), \qquad H^1\big(\mathbb{A}_X(\mathcal{O}_X(D))\big) = H^1\big(X, \mathcal{O}_X(D)\big).$$

1.2 Arithmetic Cohomology Groups

Let F be a number field with $\mathcal{O}_F$ the ring of integers. Denote by $S_{\rm fin}$, resp. S_∞, the collection of finite, resp. infinite, places of F. Write $S = S_{\rm fin} \cup S_\infty$. Let $\pi : X \to \operatorname{Spec} \mathcal{O}_F$ be an integral arithmetic variety of pure dimension $n+1$. That is, an integral Noetherian scheme X, a flat and proper morphism π with generic fiber X_F a projective variety of dimension n over F. For each $v \in S$, we write F_v the v-completion of F, and for each $\sigma \in S_\infty$, we write $X_v := X \times_{\mathcal{O}_F} \operatorname{Spec} F_v$ and write $\varphi_\sigma : X_\sigma \to X_F$ for the map induced from the natural embedding $F \hookrightarrow F_\sigma$. In particular, an arithmetic variety X consists of two parts, the finite one, which we also denote by X, and an infinite one, which we denote by X_∞. These two parts are closely interconnected.

1.2.1 Adelic rings for arithmetic surfaces

The part of our theory on arithmetic adelic complexes for finite places now becomes very simple. Indeed, our arithmetic variety X is assumed to be Noetherian, so we can apply the theory recalled in §1.1 directly. In particular, for a quasi-coherent sheaf $\mathcal{F}$ on X, we have well-defined adelic spaces

$$\mathbb{A}^{\rm fin}_{X;\,i_0,\dots,i_m}(\mathcal{F}) := \mathbb{A}_X(K_{i_0,\dots,i_m}, \mathcal{F}).$$

So to define $\mathbb{A}^{\rm ar}_{X;\,i_0,\dots,i_m}(\mathcal{F})$, we need to understand what happens on X_∞. For this purpose, we next recall Osipov-Parshin's construction of arithmetic adelic ring $\mathbb{A}^{\rm ar}_X$ for an arithmetic surface X.

Definition 5. *([OP])* **Arithmetic adelic ring of an arithmetic surface** *Let $\pi : X \to \operatorname{Spec} \mathcal{O}_F$ be an arithmetic surface, i.e., a 2-dimensional arithmetic variety, with generic fiber X_F.*
(i) Finite adelic ring: From the Parshin-Beilinson theory for the Noetherian scheme X, we define

$$\mathbb{A}^{\rm fin}_X := \mathbb{A}_{X;012}(\mathcal{O}_X) = \varinjlim_{D_1} \varprojlim_{D_2:\,D_2 \le D_1} \mathbb{A}_{X;12}(D_1)\big/\mathbb{A}_{X;12}(D_1).$$

Here D_*'s are divisors on X and $\mathbb{A}_{X;12}(D) := \mathbb{A}_{X;12}(\mathcal{O}_X(D_*))$ for $* = 1, 2$;
(ii) ∞-adelic ring: Associated to the regular integral curve X_F over F, we obtain the adelic ring

$$\mathbb{A}_{X_F} := \mathbb{A}_{X_F;01}(\mathcal{O}_{X_F}) = \varinjlim_{D_1} \varprojlim_{D_2:\,D_2 \le D_1} \mathbb{A}_{X_F;1}(D_1)\big/\mathbb{A}_{X_F;1}(D_1).$$

Here D_*'s are divisors on X_F and $\mathbb{A}_{X_F;1}(D) := \mathbb{A}_{X_F;1}(\mathcal{O}_{X_F}(D_*))$ for $* = 1, 2$.

By definition,

$$\mathbb{A}_X^\infty := \mathbb{A}_{X_F} \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} := \varinjlim_{D_1} \varprojlim_{D_2: D_2 \leq D_1} \Big(\big(\mathbb{A}_{X_F;1}(D_1)/\mathbb{A}_{X_F;1}(D_1)\big) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} \Big).$$

(iii) Arithmetic adelic ring: The arithmetic adelic ring of an arithmetic surface X is defined by

$$\mathbb{A}_X^{\rm ar} := \mathbb{A}_{X;012}^{\rm ar} := \mathbb{A}_X^{\rm fin} \bigoplus \mathbb{A}_X^\infty.$$

The essential point here is, for divisors D_i, $i = 1, 2$, over the curve X_F, when $D_2 \leq D_1$, the quotient $\mathbb{A}_{X;1}(D_1)/\mathbb{A}_{X;1}(D_1)$ is a finite dimensional F- and hence $\mathbb{Q}$-vector space.

To help the reader understand this formal definition in concrete terms, we add following examples.

Example 1. *On $X = \mathbb{P}^1_{\mathbb{Z}}$*

We have $X_{\mathbb{Q}} = \mathbb{P}^1_{\mathbb{Q}}$ and $\mathbb{Q}\big(\mathbb{P}^1_{\mathbb{Q}}\big) = \mathbb{Q}(t)$. Easily,

$$\mathbb{Q}((t)) \otimes_{\mathbb{Q}} \mathbb{R} \neq \mathbb{R}((t)).$$

However, since $\mathbb{Q}((t)) = \varinjlim_{n} \varprojlim_{m:\, m \leq n} t^{-n}\mathbb{Q}[[t]] \,/\, t^{-m}\mathbb{Q}[[t]]$ and the $\mathbb{Q}$-vector spaces $t^{-n}\mathbb{Q}[[t]] \,/\, t^{-m}\mathbb{Q}[[t]]$ are finite dimensional, we have

$$\begin{aligned}\mathbb{Q}((t)) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} =& \varinjlim_{n} \varprojlim_{m:\, m \leq n} \big(t^{-n}\mathbb{Q}[[t]]/t^{-m}\mathbb{Q}[[t]]\big) \otimes_{\mathbb{Q}} \mathbb{R} \\ =& \varinjlim_{n} \varprojlim_{m:\, m \leq n} \big(t^{-n}\mathbb{R}[[t]]/t^{-m}\mathbb{R}[[t]]\big) = \mathbb{R}((t)).\end{aligned}$$

Example 2. *Over an arithmetic surface X*

For a complete flag (X, C, x) on X (with C an irreducible curve on X and x a close point on C), let $k(X)_{C,x}$ its associated local ring. By Theorem 1, $k(X)_{C,x}$ is a direct sum of two dimensional local fields. Denote by π_C the local parameter defined by C in X. Then

$$\begin{aligned}\mathbb{A}_X^{\rm fin} =& \mathbb{A}_{X,012} = \prod_{x \in C}{}' k(X)_{C,x} := \prod_{C}{}' \Big(\prod{}'_{x: x \in C} k(X)_{C,x} \Big) \\ :=& \Big\{ \big(\sum_{i_C = -\infty}^{\infty} h_C(a_{i_C}) \pi_C^{i_C} \big)_C \in \prod_C \big(\prod_{x: x \in C} k(X)_{C,x} \big) : \\ & a_{i_C} \in \mathbb{A}_{C,01},\ a_{i_C} = 0\ (i_C \ll 0);\ \min\{i_C : a_{i_C} \neq 0\} \geq 0\ (\forall' C) \Big\},\end{aligned}$$

where h_C is a lifting defined in [MZ], which we call the Madunts-Zhukov lifting. For details, please see §3.1.2.

1.2.2 Adelic spaces at infinity

Now we are ready to treat adelic spaces at infinite places for general arithmetic varieties. Motivated by the discussion above, we make the following

Definition 6. *Let $\pi : X \to \operatorname{Spec} \mathcal{O}_F$ be an arithmetic variety. Let $S(X_F)$ be the simplicial set associated to its generic fiber X_F and $K \subset S(X_F)_m$, $m \geq 0$, a subset.*
(i) Let $\mathcal{G}$ be a coherent sheaf on X_F. We define the associated adelic spaces by

$$\mathbb{A}^\infty(K, \mathcal{G}) := \begin{cases} \prod_{p \in K} \lim_{\substack{\longleftarrow \\ l}} \big(\mathcal{G}_p / \mathfrak{m}_p^l \mathcal{G}_p \widehat{\otimes}_{\mathbb{Q}} \mathbb{R}\big), & m = 0 \\ \prod_{p \in P(X)} \lim_{\substack{\longleftarrow \\ l}} \mathbb{A}^\infty\big({}_pK, [\mathcal{G}_p / \mathfrak{m}_p^l \mathcal{G}_p]_p\big), & m > 0. \end{cases}$$

(ii) Let $\{\mathcal{G}_i\}_i$ be an inductive system of coherent sheaves on X_F and $\mathcal{F} = \lim_{\substack{\longrightarrow \\ i}} \mathcal{G}_i$. Then we define

$$\mathbb{A}^\infty(K, \mathcal{F}) := \lim_{\substack{\longrightarrow \\ i}} \mathbb{A}^\infty(K, \mathcal{G}_i).$$

Clearly, the essential part of this definition is the one for $m = 0$. Moreover, if $\mathcal{F} = \lim_{\substack{\longrightarrow \\ i}} \mathcal{G}'_i$ is another inductive limit of coherent sheaves, we have $\lim_{\substack{\longrightarrow \\ i}} \mathbb{A}^\infty(K, \mathcal{G}'_i) \simeq \lim_{\substack{\longrightarrow \\ i}} \mathbb{A}^\infty(K, \mathcal{G}_i)$, by the universal property of inductive limits since $\mathcal{G}_p / \mathfrak{m}_p^l \mathcal{G}_p$'s are $\mathbb{Q}$-vector spaces. Therefore, $\mathbb{A}^\infty(K, \mathcal{F})$ is well-defined for all quasi-coherent sheaves $\mathcal{F}$ on X_F. Moreover, as a functor from the category of coherent sheaves on X_F to that of $\mathbb{Q}$-vector spaces, $\mathbb{A}^\infty(K, *)$ is additive and exact. Hence, by [H, §1.2], $\mathbb{A}^\infty(K, *)$ commutes with the direct limits, even in general, for an inductive system $\{\mathcal{F}_i\}_i$ of quasi-coherent sheaves $\lim_{\substack{\longrightarrow \\ i}} \mathbb{A}^\infty(K, \mathcal{F}_i) \neq A^\infty(K, \lim_{\substack{\longrightarrow \\ i}} \mathcal{F}_i)$.

1.2.3 Arithmetic adelic complexes

As mentioned at the beginning of this section, for arithmetic varieties, the finite and infinite parts are closely interconnected. Therefore, when developing an arithmetic cohomology theory, we will treat them as an unify one using an uniformity condition.

Let X be an arithmetic variety with generic fiber X_F. For a point P of X_F, denote its associated Zariski closure in X by E_P. We call a flag $\delta = (\mathfrak{p}_0, \mathfrak{p}_1, \dots, \mathfrak{p}_k) \in S(X)$ *horizontal*, if there exists a flag $\delta_F = (P_0, P_1, \dots, P_k) \in S(X_F)$ such that $(\overline{\mathfrak{p}}_0, \overline{\mathfrak{p}}_1, \dots, \overline{\mathfrak{p}}_k) = (E_{P_0}, E_{P_1}, \dots, E_{P_k})$. Accordingly, for $K \subset S(X)$, we denote K^{h} the collection of all horizontal flags in K and $K^{\mathrm{nh}} = K \smallsetminus K^{\mathrm{h}}$. Simply put, our uniformity condition is a constrain on adelic components associated to horizontal flags.

Let $\mathcal{F}$ be a quasi-coherent sheaf on X, denote its induced sheaf on the generic fiber X_F by $\mathcal{F}_F$. It is well-known that $\mathcal{F}_F$ is quasi-coherent as well. Motivated by [W], we introduce the following

Definition 7. *Let X be an arithmetic variety of dimension $n+1$ and $\mathcal{F}$ a quasi-coherent sheaf on X. Fix an index tuple $(i_0, \dots, i_m)$ satisfying $i_0 \leq \dots \leq i_m$.*
(i) The finite, resp. infinite, adelic space of type $(i_0, \dots, i_m)$ associated to $\mathcal{F}$ is defined by

$$\mathbb{A}^{\mathrm{fin}}_{X;\, i_0,\dots,i_m}(\mathcal{F}) := \mathbb{A}_X\big(K_{X;\, i_0,\dots,i_m}, \mathcal{F}\big) = \mathbb{A}^{\mathrm{fin}}_X\big(K^{\mathrm{nh}}_{X;\, i_0,\dots,i_m}, \mathcal{F}\big) \oplus \mathbb{A}^{\mathrm{fin}}_X\big(K^{\mathrm{h}}_{X;\, i_0,\dots,i_m}, \mathcal{F}\big),$$

$$\textit{resp.} \qquad \mathbb{A}^\infty_{X;\, i_0,\dots,i_m}(\mathcal{F}) := \mathbb{A}^\infty_X\big(K_{X_F;\, i_0,\dots,i_m}, \mathcal{F}_F\big).$$

Here, for $Z \subset X$ or X_F, we set

$$K_{Z;i_0,\ldots,i_m} := \big\{(p_0,\ldots,p_m)\in S(Z)_m \,\big|\, \mathrm{codim}_Z\,\overline{\{p_r\}} = i_r\ \forall\, 0\le t\le m\big\};$$

(ii) The arithmetic adelic space of type $(i_0,\ldots,i_m)$ associated to $\mathcal{F}$ is defined by

$$\mathbb{A}^{\mathrm{ar}}_{X;\,i_0,\ldots,i_m}(\mathcal{F}) \ =: \begin{cases} \mathbb{A}^{\mathrm{fin}}_{X;\,i_0,\ldots,i_m}(\mathcal{F}) \bigoplus \mathbb{A}^{\infty}_{X;\,i_0,\ldots,i_{m-1}}(\mathcal{F}_F), & i_m = n+1;\\ \mathbb{A}^{\mathrm{fin}}_X\big(K^{\mathrm{nh}}_{X;\,i_0,\ldots,i_m},\mathcal{F}\big) \oplus \mathbb{A}^{\mathrm{fin,inf}}_X\big(K^{\mathrm{h}}_{X;\,i_0,\ldots,i_m},\mathcal{F}\big), & i_m \neq n+1\end{cases}$$

where

$$\mathbb{A}^{\mathrm{fin,inf}}_X\big(K^{\mathrm{h}}_{X;\,i_0,\ldots,i_m},\mathcal{F}\big) \subset \mathbb{A}^{\mathrm{fin}}_X\big(K^{\mathrm{h}}_{X;\,i_0,\ldots,i_m},\mathcal{F}\big) \bigoplus \mathbb{A}^{\infty}_{X;\,i_0,\ldots,i_m}(\mathcal{F}_F)$$

consisting of adeles satisfying, for all flags $(\mathfrak{p}_{i_0},\mathfrak{p}_{i_1},\ldots,\mathfrak{p}_{i_m})\in K_{X_F;\,i_0,\ldots,i_m}$,

$$f_{E_{\mathfrak{p}_{i_0}},E_{\mathfrak{p}_{i_1}},\ldots,E_{\mathfrak{p}_{i_m}}} = f_{\mathfrak{p}_{i_0},\mathfrak{p}_{i_1},\ldots,\mathfrak{p}_{i_m}};$$

(iii) For $m\ge 0$, define the m-th reduced arithmetic adelic space $\mathbb{A}^{\mathrm{ar}}_{X;\,m}(\mathcal{F})$ of $\mathcal{F}$ by

$$\mathbb{A}^m_{\mathrm{ar,\,red}}(X,\mathcal{F}) := \bigoplus_{\substack{(i_0,\ldots,i_m)\\ 0\le i_0<i_1<\cdots<i_m\le n+1}} \mathbb{A}^{\mathrm{ar}}_{X;\,i_0,\ldots,i_m}(\mathcal{F}).$$

Remarks. (i) For any $\mathfrak{p}\in P(X_F)$, $\mathcal{O}_{X,E_\mathfrak{p}} = \mathcal{O}_{X_F,\mathfrak{p}}$ and $k(X)_{E_\mathfrak{p}} = k(X_F)_\mathfrak{p}$. Consequently, for any $(\mathfrak{p}_0,\ldots,\mathfrak{p}_m)\in S(X_F)_m$, we have a natural morphism

$$\mathbb{A}\big((E_{\mathfrak{p}_0},\ldots,E_{\mathfrak{p}_m}),\mathcal{F}\big) = \mathbb{A}\big((\mathfrak{p}_0,\ldots,\mathfrak{p}_m),\mathcal{F}_F\big).$$

since $\mathcal{F}$ is quasi-coherent. It is in this sense we use the relation $f_{E_{\mathfrak{p}_0},E_{\mathfrak{p}_1},\ldots,E_{\mathfrak{p}_m}} = f_{\mathfrak{p}_0,\mathfrak{p}_1,\ldots,\mathfrak{p}_m}$ above. (In particular, if $\mathfrak{p}_i$'s are vertical, there are no conditions on the corresponding components.) Clearly, this uniformity condition is an essential one, since it characters the natural interconnection between finite and infinite components of arithmetic adelic elements.
(ii) In part (ii) of the definition, we need the space $\mathbb{A}^{\infty}_{X;\,\emptyset}(\mathcal{F}_F)$. Here, to complete our definition, for an arithmetic variety X, we view $\mathbb{A}^{\infty}_{X;\,\emptyset}(\mathcal{F}_F)$ as the (-1)-level of the adelic complex for its generic fiber X_F. That is to say, we define it as follows. By [Y, p. 63], we have the (-1)-simplex $\underline{1}_U$ for open $U\subset X$. Set then $S(X_F)_{-1} = \{\underline{1}_U \mid U\subset X:\text{open}\}$, and, for $K\subset S(X_F)_{-1}$, let

$$\mathbb{A}^{\infty}_{X;\,\emptyset}(K,\mathcal{F}_F) := \begin{cases}\mathcal{F}_F(U_{K,F})\otimes_{\mathbb{Q}}\mathbb{R}, & \dim X\ge 2\\ \{\, s_F\in\mathcal{F}_F(U_{K,F})\otimes\mathbb{R} \mid s\in\mathcal{F}(U_K)\,\}, & \dim X = 1\end{cases}$$

where $U_K := \cup_{\underline{1}_U\in K} U$ and s_F denotes the section induced by s. The reason for separation of arithmetic curves with others in this latest definition is that arithmetic varieties are relative over arithmetic curves.

Moreover, from standard homotopy theory, if we introduce the boundary morphisms by

$$\begin{array}{rccc} d_i^m: & \bigoplus \mathbb{A}^{\mathrm{ar}}_{X;\,l_0,\ldots,l_{m-1}}(\mathcal{F}) & \longrightarrow & \bigoplus \mathbb{A}^{\mathrm{ar}}_{X;\,k_0,\ldots,k_m}(\mathcal{F})\\ & (a_{l_0,\ldots,l_{m-1}}) & \mapsto & (a_{k_0,\ldots,\hat{k}_i,\ldots,k_m}); \end{array}$$

and $d_m = \sum_{i=0}^m (-1)^i d_i^m: \bigoplus \mathbb{A}^{\mathrm{ar}}_{X;\,k_0,\ldots,l_{m-1}}(\mathcal{F}) \longrightarrow \bigoplus \mathbb{A}^{\mathrm{ar}}_{X;\,k_0,\ldots,k_m}(\mathcal{F})$, we have

Proposition 8. $\left(\mathbb{A}^*_{\mathrm{ar,\,red}}(X,\mathcal{F}),d^*\right)$ *defines a complex of abelian groups.*

All in all, we are now ready to introduce the following

Main Definition. *Let $\pi : X \to \operatorname{Spec}\mathcal{O}_F$ be an arithmetic variety. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then we define the i-th adelic arithmetic cohomology groups of $\mathcal{F}$ by*

$$H^i_{\mathrm{ar}}(X,\mathcal{F}) := H^i\big(\mathbb{A}^*_{\mathrm{ar,\,red}}(X,\mathcal{F}),d^*\big),$$

the i-th cohomology group of the complex $\big(\mathbb{A}^*_{\mathrm{ar,\,red}}(X,\mathcal{F}),d^*\big)$.

Consequently, we have the following

Theorem 9. *If X is an arithmetic variety of dimension $n+1$, then*

$$H^i_{\mathrm{ar}}(X,\mathcal{F}) = 0 \qquad \text{unless } i = 0,1,\ldots,n+1.$$

Proof. Indeed, outside the range $0 \le i \le n+1$, the complex consists of zero.

1.2.4 Cohomology theory for arithmetic curves

We here give an example of the above theory for arithmetic curves, which was previously developed in [W], based on Tate's thesis ([T]).

Let $D = \sum_{i=1}^r n_i\mathfrak{p}_i$ be a divisor on $X = \operatorname{Spec}\mathcal{O}_F$. Write $n_i = \operatorname{ord}_{\mathfrak{p}_i}(D)$. For simplicity, we use $\mathbb{A}^*_\bullet(D)$ instead of $\mathbb{A}^*_\bullet(\mathcal{O}_X(D))$. Then, by the same calculation as in §1.1.3, we have $\mathbb{A}^{\mathrm{fin}}_{X;01}(D) = \big\{(a_p) \in \prod_{p\in X} F_p \,\big|\, a_p \in \mathcal{O}_p\ \forall' p\big\}$. And, since $D_\eta = 0$ is trivial, $\mathbb{A}^\infty_{X;0}(D) = \lim_{\leftarrow l} \mathcal{O}_{X_F,\eta}/\mathfrak{m}^l_\eta\mathcal{O}_{X_F,\eta} \otimes_{\mathbb{Q}} \mathbb{R} = \lim_{\leftarrow l}\big(F/\{0\}\big)\otimes_{\mathbb{Q}} \mathbb{R} = F\otimes_{\mathbb{Q}}\mathbb{R} = \prod_{\sigma\in S_\infty} F_\sigma$. Therefore,

$$\mathbb{A}^{\mathrm{ar}}_{X;01}\big(\mathcal{O}_X(D)\big) = \mathbb{A}^{\mathrm{fin}}_{01}(D)\oplus\mathbb{A}^\infty_0(D) = \big\{(a_\mathfrak{p}) \in \prod_{\mathfrak{p}\in S} F_\mathfrak{p} \,\big|\, a_\mathfrak{p}\in\mathcal{O}_\mathfrak{p}\ \forall'\mathfrak{p}\in S_{\mathrm{fin}}\big\}.$$

In particular, it coincides with the standard adelic ring $\mathbb{A}_F$ of F, hence is independent of D.

To understand $\mathbb{A}^{\mathrm{ar}}_0(D)$, we first calculate $\mathbb{A}^{\mathrm{fin}}_0(D)$. With the same calculation as in §1.1.3 again, we have $\mathbb{A}^{\mathrm{fin}}_{X;0}(D) = F$. Note that, from above, $\mathbb{A}^\infty_{X;0}(D) = F\otimes_{\mathbb{Q}}\mathbb{R}$. Thus, by definition,

$$\mathbb{A}^{\mathrm{ar}}_{X;0}(D) = \big\{(a_v;a_\sigma)\in\mathbb{A}^{\mathrm{fin}}_{X;0}(D)\oplus\mathbb{A}^\infty_{X;0}(D)\,\big|\,(a_v) = i_{\mathrm{fin}}(f),(a_\sigma) = i_\infty(f)\,\exists f\in F\big\}$$

is then isomorphic to F, and hence also independent of D.

From our definition, $\mathbb{A}^{\mathrm{ar}}_{X;1}(D) = \mathbb{A}^{\mathrm{fin}}_{X;1}(\mathcal{O}_X(D))\oplus\mathbb{A}^\infty_{X;\emptyset}(\mathcal{O}_X(D))$. To understand it, we first calculate $\mathbb{A}^{\mathrm{fin}}_{X;1}(D)$. With the same calculation as in §1.1.3, we have $\mathbb{A}^{\mathrm{fin}}_{X;1}(D) = \big\{(a_\mathfrak{p})\in\mathbb{A}^{\mathrm{fin}}_{X;01}\,\big|\,\operatorname{ord}_\mathfrak{p}(a_\mathfrak{p})+\operatorname{ord}_\mathfrak{p}(D)\ge 0\big\}$. Then, by definition, we have $\mathbb{A}^\infty_{X;\emptyset}(D) = \{s\in F\,|\,\operatorname{ord}_\mathfrak{p}(s)+\operatorname{ord}_\mathfrak{p}(D)\ge 0\big\}\otimes_{\mathbb{Q}}\mathbb{R}$, since $D = \sum_\mathfrak{p}\operatorname{ord}_\mathfrak{p}(D)\mathfrak{p}$, and hence if $U = X - \{\mathfrak{p}_1,\ldots,\mathfrak{p}_r\}$, $\mathcal{O}_X(U)$ is trivial.

In this way, we get the associated arithmetic adelic complex

$$0\longrightarrow\mathbb{A}^{\mathrm{ar}}_{X;0}(\mathcal{O}_X(D))\oplus\mathbb{A}^{\mathrm{ar}}_{X;1}(\mathcal{O}_X(D))\xrightarrow{d^1}\mathbb{A}^{\mathrm{ar}}_{X;01}(\mathcal{O}_X(D))\longrightarrow 0$$

is given by: $0 \to F \oplus \mathbb{A}^{\rm ar}_{X;1}(\mathcal{O}_X(D)) \xrightarrow{d^1} \mathbb{A}_F \to 0, (a_0,a_1) \mapsto a_1 - a_0$. Therefore,

$$H^0_{\rm ar}\big(F, \mathcal{O}_X(D)\big) = F \cap \mathbb{A}^{\rm ar}_{X;1}(\mathcal{O}_X(D)),$$
$$H^1_{\rm ar}\big(F, \mathcal{O}_X(D)\big) = \mathbb{A}_F \big/ \big(F + \mathbb{A}^{\rm ar}_{X;1}(\mathcal{O}_X(D))\big).$$

In fact, a complete cohomology theory is developed for arithmetic curves $\mathrm{Spec}\,\mathcal{O}_F$ in [W]. For an $\mathcal{O}_F$-lattice Λ, i.e., a motorized locally free sheaf on $\mathrm{Spec}\,\mathcal{O}_F$, we introduce the associated topological cohomology groups $H^0_{\rm ar}(F,\Lambda)$ and $H^1_{\rm ar}(F,\Lambda)$, with $H^0_{\rm ar}(F,\Lambda)$ discrete and $H^1_{\rm ar}(F,\Lambda)$ compact. Consequently, using Fourier analysis for locally compact groups, we obtain their arithmetic counts $h^0_{\rm ar}(F,\Lambda)$ and $h^1_{\rm ar}(F,\Lambda)$.

Theorem 10. Cohomology Theory for Arithmetic Curves *([W])*

Let F be a number field with $\mathcal{O}_F$ the ring of integers. Let ω_F be the Arakelov dualizing lattice of $\mathrm{Spec}\,\mathcal{O}_F$ *and Δ_F be the discriminant of F. Then, for an $\mathcal{O}_F$-lattice Λ of rank n with its dual $\Lambda^\vee$, we have*

*(1) (1.i) (***Topological Duality***) As locally compact topological groups,*

$$H^1_{\rm ar}(\widehat{F, \omega_F \otimes \Lambda^\vee}) = H^0_{\rm ar}(F,\Lambda);$$

Here $\widehat{\ }$ denotes the Pontryagin dual.

*(1.ii) (***Arithmetic Duality***)*

$$h^1_{\rm ar}(F, \omega_F \otimes \Lambda^\vee) = h^0_{\rm ar}(F,\Lambda);$$

*(2) (***Arithmetic Riemann-Roch Theorem***)*

$$h^0_{\rm ar}(F,\Lambda) - h^1_{\rm ar}(F,\Lambda) = \deg_{\rm ar}(\Lambda) - \frac{n}{2}\log|\Delta_F|.$$

*(3) (***Ampleness, Positivity and Vanishing Theorem***) The following statements are equivalent:*

(3.i) Rank one $\mathcal{O}_F$-lattice A is arithmetic positive;

(3.ii) Rank one $\mathcal{O}_F$-lattice A is arithmetic ample; and

(3.iii) For rank one $\mathcal{O}_F$-lattice A and any $\mathcal{O}_F$-lattice L,

$$\lim_{n\to\infty} h^1_{\rm ar}(F, A^n \otimes L) = 0.$$

*(4) (***Effective Vanishing Theorem***) Assume that Λ is a semi-stable $\mathcal{O}_F$-lattice satisfying* $\deg_{\rm ar}(\Lambda) \leq -[F:\mathbb{Q}]\cdot \frac{n\log n}{2}$*, then we have*

$$h^0_{\rm ar}(F,\Lambda) \leq \frac{3^{n\,[F:\mathbb{Q}]}}{1-\log 3/\pi}\cdot \exp\Big(-\pi[F:\mathbb{Q}]\cdot e^{-\frac{\deg_{\rm ar}(L)}{n}}\Big).$$

For details, please refer to [W].

2 Arithmetic Surfaces

In the sequel, by an arithmetic surface, we mean a 2-dimensional regular integral Noetherian scheme X together with a flat, proper morphism $\pi : X \to \operatorname{Spec} \mathcal{O}_F$. Here $\mathcal{O}_F$ denotes the ring of integers of a number field F. In particular, the generic fiber X_F is a geometrically connected, regular, integral projective curve defined over F.

2.1 Local Residue Pairings

Theory of residues for arithmetic surfaces, as a special case of Grothendieck's residue theory, can be realized using Kähler differentials as done in [L, Ch III, §4]. However, here we follow ([M1,2]) to give a rather precise realization in terms of structures of two dimensional local fields.

2.1.1 Residue maps for local fields

(A) Continuous differentials

Let $(A, \mathbf{m}_A)$ be a local Noetherian ring and N an A-module N. Denote by N^{sep} the maximal Hausdorff quotient of N for the $\mathbf{m}_A$-adic topology, i.e., $N^{\text{sep}} = N / \bigcap_{n=1}^{\infty} \mathbf{m}_A^n N$. In particular, if A is an R-algebra for a certain ring R, then we have the differential module $\Omega_{A/R}$ and hence $\Omega_{A/R}^{\text{sep}}$. Thus, if F is a complete discrete valuation field and K a subfield such that $\operatorname{Frac}(K \cap \mathcal{O}_F) = K$, then we have the space of the continuous differentials

$$\Omega_{F/K}^{\text{cts}} := \Omega_{\mathcal{O}_F/\mathcal{O}_F \cap K}^{\text{sep}} \otimes_{\mathcal{O}_F} F.$$

Consequently, if F'/F is a finite, separable field extension, then $\Omega_{F'/K}^{\text{cts}} = \Omega_{F/K}^{\text{cts}} \otimes_F F'$ and hence there is a natural trace map $\operatorname{Tr}_{F'/F} : \Omega_{F'/K}^{\text{cts}} \longrightarrow \Omega_{F/K}^{\text{cts}}$.

(B) Equal characteristic zero

Let F be a two-dimensional local field of equal characteristic zero. Then F contains a unique subfield k_F of coefficients, up to isomorphism, such that $F \simeq k_F((t))$ for a suitable uniformizer t. In particular, $\Omega_{\mathcal{O}_F/\mathcal{O}_{k_F}}^{\text{sep}} \simeq \mathcal{O}_F \cdot dt$ is a free $\mathcal{O}_F$-module of rank one. We define the residue map for F by

$$\operatorname{res}_F : \Omega_{F/k_F}^{\text{cts}} \longrightarrow k_F, \qquad \omega = f\,dt \mapsto \operatorname{coeft}_{t^{-1}}(f).$$

By [M1], this is well defined, i.e., independent of the choice of t. Moreover, for a finite field extension F'/F, we have the following commutative diagram

$$\begin{array}{ccc} \Omega_{F'/k_F}^{\text{cts}} & \xrightarrow{\operatorname{res}_{F'}} & k_{F'} \\ \operatorname{Tr}_{F'/F} \downarrow & & \downarrow \operatorname{Tr}_{k_{F'}/k_F} \\ \Omega_{F/k_F}^{\text{cts}} & \xrightarrow{\operatorname{res}_F} & k_F. \end{array}$$

(C) Mixed characteristic

Let L be a two dimensional local field of mixed characteristics. Then the constant field k_L of L coincides with the algebraic closure of $\mathbb{Q}_p$ within L for a certain prime number p, and L itself is a finite field extension over $k_L\{\{t\}\}$ for a certain uniformizer t. Here, by definition,

$$k_L\{\{t\}\} := \Big\{ \sum_{i=-\infty}^{\infty} a_i t^i \,:\, a_i \in k_L,\ \inf_i \big\{\nu_{k_L}(a_i)\big\} > -\infty\ (\forall i),\quad a_i \to 0\ (i \to -\infty)\Big\}.$$

Moreover, by [M1], $\Omega^{\mathrm{sep}}_{\mathcal{O}_{k_L\{\{t\}\}/\mathcal{O}_{k_L}}} = \mathcal{O}_{k_L\{\{t\}\}}\, dt \bigoplus \mathrm{Tors}\big(\Omega^{\mathrm{sep}}_{\mathcal{O}_{k_L\{\{t\}\}/\mathcal{O}_{k_L}}}\big)$. We define the residue map, first, for $k_L\{\{t\}\}$, by

$$\mathrm{res}_{k_L\{\{t\}\}} : \Omega^{\mathrm{cts}}_{k_L\{\{t\}\}/k_L} \longrightarrow k_L, \qquad \omega = f\,dt \mapsto -\mathrm{coeft}_{t^{-1}}(f);$$

then, for L, by the composition

$$\mathrm{res}_L : \Omega^{\mathrm{cts}}_{L/k_L} \xrightarrow{\mathrm{Tr}_{L/k_L\{\{t\}\}}} \Omega^{\mathrm{cts}}_{k_L\{\{t\}\}/k_L} \xrightarrow{\mathrm{res}_{k_L\{\{t\}\}}} k_L.$$

This is well defined by [M1]. Consequently, if L'/L is a finite field extension, we have the commutative diagram

$$\begin{array}{ccc} \Omega^{\mathrm{cts}}_{L'/k_L} & \xrightarrow{\mathrm{res}_{L'}} & k_{L'} \\ \mathrm{Tr}_{L'/L}\downarrow & & \downarrow \mathrm{Tr}_{k_{L'}/k_L} \\ \Omega^{\mathrm{cts}}_{L/k_L} & \xrightarrow{\mathrm{res}_L} & k_L. \end{array}$$

2.1.2 Local residue maps

As above, let F be a number field and $\pi : X \to \mathrm{Spec}\,\mathcal{O}_F$ be an arithmetic surface with X_F its generic fiber.

For each closed point $x \in X$, and a prime divisor C on X with $x \in C$, by Theorem 1, the local ring $k(X)_{C,x}$ is a finite direct sum of two dimensional local fields, i.e.,

$$k(X)_{C,x} = \bigoplus k(X)_{C_i,x},$$

where C_i's are normalized branches of the curve C in a formal neighborhood U of x. Set then

$$\mathrm{res}_{C,x} = \sum_i \mathrm{res}_{k(X)_{C_i,x}},$$

which takes the values in $F_{\pi(x)}$, the local field of F at the place $\pi(x)$. Recall also that, following [T], we have the canonical character

$$\lambda_{\pi(x)} : F_{\pi(x)} \xrightarrow{\mathrm{Tr}_{F_{\pi(x)}/\mathbb{Q}_p}} \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1.$$

Introduce accordingly

$$\mathrm{Res}_{C,x} := \lambda_{\pi(x)} \circ \mathrm{res}_{C,x}.$$

On the other hand, for each closed point $P \in X_F$,

$$k(X_F)_P \widehat{\bigotimes}_F \Big(\prod_{\sigma\in S_\infty} F_\sigma\Big) = \Big(\bigoplus \mathbb{R}((t))\Big) \bigoplus \Big(\bigoplus \mathbb{C}((t))\Big)$$

is a finite direct sum of local fields $\mathbb{R}((t))$ and $\mathbb{C}((t))$. Hence, similarly, for each $\sigma \in S_\infty$, we have the associated residue maps $\mathrm{res}_{P,\sigma}$. Define

$$\mathrm{Res}_{P,\sigma} = \lambda_\sigma \circ \mathrm{res}_{P,\sigma}.$$

Here, as in [T], to make all compatible, we set $\lambda_\sigma(x) = -\mathrm{Tr}_{F_\sigma/\mathbb{R}}$, i.e., with a minus sign added.

2.2 Global Residue Pairing

The purpose here is to introduce a non-degenerate global residue pairing on the arithmetic adelic ring of an arithmetic surface.

2.2.1 Global residue pairing

Let $\pi : X \to \operatorname{Spec} \mathcal{O}_F$ be an arithmetic surface with X_F its generic fiber. Then, by §1.2.1, we have the associated arithmetic adelic ring

$$\mathbb{A}_X^{\mathrm{ar}} := \mathbb{A}_{X,012}^{\mathrm{ar}} := \mathbb{A}_X^{\mathrm{fin}} \bigoplus \mathbb{A}_X^{\infty}$$

with $\mathbb{A}_X^{\mathrm{fin}} = \mathbb{A}_{X,012}$ the adelic ring for the 2-dimensional Noetherian scheme X and $\mathbb{A}_X^{\infty} := \mathbb{A}_{X_F} \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} := \varinjlim_{D_1} \varprojlim_{D_2 \leq D_1} \Big(\big(\mathbb{A}_{X_F}(D_1)/\mathbb{A}_{X_F}(D_2)\big) \bigotimes_F \prod_{\sigma \in S_\infty} F_\sigma \Big)$. By an abuse of notation, we will write elements of $\mathbb{A}_X^{\mathrm{fin}}$ as $(f_{C,x})_{C,x}$, or even $(f_{C,x})$, and elements of $\mathbb{A}_X^{\infty}$ as $(f_P)_P$, or even (f_P).

Fix a rational differential $\omega = f(t)\,dt \not\equiv 0$ on X. Then, we define a global pairing with respect to ω by

$$\begin{array}{cccc} \langle \cdot, \cdot \rangle_\omega : & \mathbb{A}_X^{\mathrm{ar}} \times \mathbb{A}_X^{\mathrm{ar}} & \longrightarrow & \mathbb{S}^1 \\ & \big((f_{C,x}, f_{P,\sigma}), (g_{C,x}, g_{P,\sigma})\big) & \mapsto & \sum_{C \subset X, x \in C:\, \pi(x) \in S_{\mathrm{fin}}} \mathrm{Res}_{C,x}(f_{C,x} g_{C,x} \omega) \\ & & & + \sum_{P \in X_F} \sum_{\sigma \in S_\infty} \mathrm{Res}_{P,\sigma}(f_{P,\sigma} g_{P,\sigma} \omega). \end{array}$$

Lemma 11. *Let X be an arithmetic surface and ω is a non-zero rational differential on X. Then the global pairing with respect to ω above is well defined.*

Proof. Write $\sum_{C \subset X, x \in C:\, \pi(x) \in S_{\mathrm{fin}}} \mathrm{Res}_{C,x}(f_{C,x} g_{C,x} \omega)$ as a double summations $\sum_{C \subset X} \sum_{x \in C:\, \pi(x) \in S_{\mathrm{fin}}} \mathrm{Res}_{C,x}(f_{C,x} g_{C,x} \omega)$. Then, by Example 2, for all but finitely many curves C, $\mathrm{Res}_{C,x}(f_{C,x} g_{C,x} \omega) = 0$. So it suffices to show that for a fixed curve C, $\sum_{x \in C:\, \pi(x) \in S_{\mathrm{fin}}} \mathrm{Res}_{C,x}(f_{C,x} g_{C,x} \omega)$ is finite. This is a direct consequence of the definition of $\mathbb{A}_{C,01}$ and $\mathbb{A}_X^{\mathrm{fin}}$ in Example 2.

2.2.2 Non-degeneracy

Proposition 12. *The residue pairing $\langle \cdot, \cdot \rangle_\omega$ on $\mathbb{A}_X^{\mathrm{ar}}$ is non-degenerate.*

Proof. Let $g \in \mathbb{A}_X^{\mathrm{ar}}$ be an adelic element such that, for all $f \in \mathbb{A}_X^{\mathrm{ar}}$, $\langle f, g \rangle_\omega = 0$. We show that $g = 0$.

Rewrite the summation in the definition of $\langle \cdot, \cdot \rangle_\omega$ according to prime horizontal curves E_P^{ar} associated to closed points $P \in X_F$ and prime vertical curves $V \subset X$ appeared in the fibers of π. Namely, $\sum_{P \in X_F} \sum_{x \in E_P^{\mathrm{ar}}} + \sum_{V \in X} \sum_{x \in V}$. Then, note that, for a fixed adelic element $g = (g_{C,x}, g_{P,\sigma}) \in \mathbb{A}_X^{\mathrm{ar}}$,

$$\langle (f_{C,x}, f_{P,\sigma}), (g_{C,x}, g_{P,\sigma}) \rangle = 0 \qquad \forall (f_{C,x}, f_{P,\sigma}) \in \mathbb{A}_X^{\mathrm{ar}}.$$

We have

$$\sum_{P \in X_F} \sum_{x \in E_P^{\mathrm{ar}}} \mathrm{Res}_{E_P,x}(g_{E_P,x} f \omega) + \sum_{V \in X} \sum_{x \in V} \mathrm{Res}_{E_P,x}(g_{E_P,x} f \omega) = 0 \qquad \forall f \in k(X).$$

Now assume, otherwise, that $g \neq 0$. There exists either some vertical curve C such that $0 \neq g_{C,x} \in k(X)_{C,x}$, or, a certain algebraic point P of the generic fiber X_F such that $0 \neq g_{P,\sigma} \in k(X_F)_P \hat{\otimes}_F F_\sigma$. In case $g_{C,x} \neq 0$, by definition, $g_{C,x} = (g_{C_i,x}) \in \bigoplus_i k(X)_{C_i,x} = k(X)_{C,x}$, where C_i runs over all branches of C, and $k(X)_{C_i,x}$ is a two-dimensional local field. Fix a branch C_{i_0} such that $g_{C_{i_0},x} \neq 0$. By definition, $\mathrm{Res}_{C_i,x} := \lambda_{\pi(x)} \circ \mathrm{res}_{C_i,x}$. So we can choose an element $h \in k(X)_{C_{i_0},x}$ such that $\mathrm{Res}_{C_{i_0},x}(h\omega) \neq 0$. For such h, we then take $f_{C_{i_0},x} \in k(X)_{C_{i_0},x}$ such that $f_{C_{i_0},x} g_{C_{i_0},x} = h$. Accordingly, if we construct an adelic element f by taking all other components $f_{C_i,x}$ to be zero but the $f_{C_{i_0},x}$, then we have $\langle f, g\rangle_\omega = \mathrm{Res}_{C_{i_0},x}(f_{C_{i_0},x} g_{C_{i_0},x}\omega) = \mathrm{Res}_{C_{i_0},x}(h\omega) \neq 0$. This contradicts to our original assumption that $\langle f, g\rangle_\omega = 0$. Hence, all the components of g corresponding to vertical curves are zero. Since we can use the same argument for the components of g corresponding to algebraic points of X_F to conclude that all the related components are zero as well, so $g = 0$. This completes the proof.

2.3 Adelic Subspaces

2.3.1 Level two subspaces

Let $\pi : X \to \mathrm{Spec}\, \mathcal{O}_F$ be an arithmetic surface. Our purpose here is to introduce certain level two intrinsic subspaces of $\mathbb{A}_X^{\mathrm{ar}} := \mathbb{A}_{X,012}^{\mathrm{ar}} := \mathbb{A}_X^{\mathrm{fin}} \oplus \mathbb{A}_X^{\infty}$.

To start with, we analyze the structures of $\mathbb{A}_{X,01}^{\mathrm{fin}}$, one of the level two subspaces of $\mathbb{A}_X^{\mathrm{fin}} = \mathbb{A}_{X,012}^{\mathrm{fin}}$. By definition, see e.g., [P1], an element $(f_{C,x})_{C,x} \in \mathbb{A}_X^{\mathrm{fin}}$ belongs to $\mathbb{A}_{X,01}^{\mathrm{fin}}$, if, for all curves C, the partial components $(f_{C,x})_{x\in C}$ are independent of x. So we may simply write elements of $\mathbb{A}_{X,01}^{\mathrm{fin}}$ as $(f_C)_C$. On the other hand, with respect to π, curves on X may be classified as being either vertical or horizontal. Therefore, we may and will write $(f_C)_C = (f_C)_{C:\,\mathrm{ver}} \times (f_C)_{C:\,\mathrm{hor}}$. Accordingly, we set $\mathbb{A}_{X,01}^{\mathrm{fin}} = \mathbb{A}_{X,01}^{\mathrm{fin,v}} \oplus \mathbb{A}_{X,01}^{\mathrm{fin,h}}$, where $\mathbb{A}_{X,01}^{\mathrm{fin,v}}$, resp., $\mathbb{A}_{X,01}^{\mathrm{fin,h}}$ denotes the collections of $(f_C)_{C:\,\mathrm{ver}}$, resp., $(f_C)_{C:\,\mathrm{hor}}$. Furthermore, if C is horizontal, there exists an algebraic point P of X_F such that $C = \overline{\{P\}}^X$, the Zariski closure of P in X. For simplicity, write $C = E_P$. Then $f_{E_P} \in F(X)_{X_P}$. But $F(X)_{E_P} = F(X_F)_P$. So it makes sense for us to talk about whether $f_{E_P} = f_P$ for a certain element $f_P \in F(X_F)_P$.

Fix a Weil divisor $D = D_v + D_h$ on X, where $D_v = \sum_F n_V V$ with V irreducible vertical curves and $D_h = \sum_P n_P E_P$ with E_P the horizontal curves. In particular, D induces a divisor $D_F = \sum_P n_P P$ on X_F. Following the uniformity condition in Definition 7(ii) (and that for arithmetic curves recalled in §1.2.4), we introduce level two intrinsic subspaces $\mathbb{A}_{X,01}^{\mathrm{ar}}$, $\mathbb{A}_{X,02}^{\mathrm{ar}}$, $\mathbb{A}_{X,12}^{\mathrm{ar}}(D)$ by

$$\mathbb{A}_{X,01}^{\mathrm{ar}} = \big\{(f_{C,x}) \times (f_P) \in \mathbb{A}_X^{\mathrm{ar}} \,\big|\, (f_{C,x})_{C,x} = (f_C)_{C,x} \in \mathbb{A}_{X,01}^{\mathrm{fin}},\ f_{E_P} = f_P\ \forall P \in X_F\big\},$$

$$\mathbb{A}_{X,02}^{\mathrm{ar}} = \mathbb{A}_{X,02}^{\mathrm{fin}} \bigoplus k(X_F) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} \quad \text{and} \quad \mathbb{A}_{X,12}^{\mathrm{ar}}(D) := \mathbb{A}_{X,12}^{\mathrm{fin}}(D) \bigoplus \big(\mathbb{A}_{X_F}(D_F) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R}\big),$$

where
$$\mathbb{A}_{X,12}^{\mathrm{fin}}(D) := \big\{(f_{C,x}) \in \mathbb{A}_X^{\mathrm{fin}} \,\big|\, \mathrm{ord}_C(f_{C,x}) + \mathrm{ord}_C(D) \geq 0\ \forall C \subset X\big\},$$
$$\mathbb{A}_{X_F}(D_F) := \big\{(f_P) \in \mathbb{A}_{X_F} \,\big|\, \mathrm{ord}_P(f_P) + \mathrm{ord}_P(D_F) \geq 0\ \forall P \in X_F\big\},$$
$$\mathbb{A}_{X_F}(D_F) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} := \lim_{\longrightarrow D'_F : D'_F \leq D_F} \Big(\mathbb{A}_{X_F}(D_F)/\mathbb{A}_{X_F}(D'_F) \bigotimes_{\mathbb{Q}} \mathbb{R}\Big).$$

Here we have used the natural imbedding $k(X) = k(X_F) \hookrightarrow \mathbb{A}_{X_F} \hookrightarrow \mathbb{A}_X^{\rm ar}$.

Accordingly, we then also obtain three level one subspaces

$$\begin{aligned} \mathbb{A}_{X,0}^{\rm ar} :=& \mathbb{A}_{X,01}^{\rm ar} \cap \mathbb{A}_{X,02}^{\rm ar}, \qquad\qquad \text{and} \\ \mathbb{A}_{X,1}^{\rm ar}(D) :=& \mathbb{A}_{X,01}^{\rm ar} \cap \mathbb{A}_{X,12}^{\rm ar}(D), \qquad \mathbb{A}_{X,2}^{\rm ar}(D) := \mathbb{A}_{X,02}^{\rm ar} \cap \mathbb{A}_{X,12}^{\rm ar}(D). \end{aligned}$$

Lemma 13. *Let X be an arithmetic surface, D be a Weil divisor on X with D_F its induced divisor on X_F. We have*

(i) $\mathbb{A}_{X,0}^{\rm ar} = k(X)$;

(ii) $\mathbb{A}_{X,1}^{\rm ar}(D) = \{(f_C)_{C,x} \times (f_P) \in \mathbb{A}_X^{\rm ar} : (f_C)_{C,x} \in \mathbb{A}_{X,1}(D),\ f_{E_P} = f_P\ \forall P \in X_F\}$;

(iii) $\mathbb{A}_{X,2}^{\rm ar}(D) = \{(f_x)_{C,x} \times (f) \in \mathbb{A}_X^{\rm ar} :$
$(f_x)_{C,x} \in \mathbb{A}_{X,2}(D),\ f \in \mathbb{A}_{X_F}(D_F) \cap k(X_F) = H^0(X_F, D_F) \otimes_{\mathbb{Q}} \mathbb{R}\}$

(iv) Under the natural boundary map, $(\mathbb{A}_{\rm ar}^*(X,D), d^*)$:

$$0 \to \mathbb{A}_{X,0}^{\rm ar} \oplus \mathbb{A}_{X,1}^{\rm ar}(D) \oplus \mathbb{A}_{X,2}^{\rm ar}(D) \to \mathbb{A}_{X,01}^{\rm ar} \oplus \mathbb{A}_{X,02}^{\rm ar} \oplus \mathbb{A}_{X,12}^{\rm ar}(D) \to \mathbb{A}_{X,012}^{\rm ar} \to 0$$

forms a complex, the adelic complex for D.

Proof. The first three are direct consequences of the construction, while (iv) is standard from homotopy theory. Indeed, we only need to write down the boundary maps: the first is the diagonal embedding, the second is given by $(x_0, x_1, x_2) \mapsto (x_0 - x_1, x_1 - x_2, x_2 - x_0)$ and the final one is given by $(x_{01}, x_{02}, x_{12}) \mapsto x_{01} + x_{02} - x_{12}$.

As a direct consequence of (ii) and (iii), we have the following

Corollary 14. *Let X be an arithmetic surface, D be a Weil divisor on X. Then, we have the following induced ind-pro structures on $\mathbb{A}_{X,01}^{\rm ar}$ and $\mathbb{A}_{X,02}^{\rm ar}$:*

$$\begin{aligned} \mathbb{A}_{X,01}^{\rm ar} &= \varinjlim_{D'} \varprojlim_{\substack{D' \\ D' \le D}} \mathbb{A}_{X,1}^{\rm ar}(D) / \mathbb{A}_{X,1}^{\rm ar}(D'), \\ \mathbb{A}_{X,02}^{\rm ar} &= \varinjlim_{D'} \varprojlim_{\substack{D' \\ D' \le D}} \mathbb{A}_{X,2}^{\rm ar}(D) / \mathbb{A}_{X,2}^{\rm ar}(D'). \end{aligned}$$

Our definitions here are specializations of definitions in §1.2.3 for $\mathcal{F} = \mathcal{O}_X(D)$ over arithmetic surface X. We leave the details to the reader.

2.3.2 Perpendicular subspaces

For late use, we here establish a fundamental property for the level two arithmetic adelic subspaces introduced above. In fact, as we will see below, this property, in turn, characterizes these subspaces.

Fix a non-zero rational differential ω on the arithmetic surface X. Then, by Proposition 12, we have a natural non-degenerate pairing $\langle \cdot, \cdot \rangle_\omega$ on $\mathbb{A}_X^{\rm ar}$. For a subspace V of $\mathbb{A}_X^{\rm ar}$, set

$$V^\perp := \left\{ w \in \mathbb{A}_X^{\rm ar} \,\middle|\, \langle w, v \rangle_\omega = 0 \ \ \forall v \in V \right\}$$

be its perpendicular subspace of V in $\mathbb{A}_X^{\rm ar}$ with respect to $\langle \cdot, \cdot \rangle_\omega$. Then, we have the following important

Proposition 15. *Let X be an arithmetic surface, D be a Weil divisor and ω be a non-zero rational differential on X. Denote by (ω) the divisor on X associated to ω. Then we have*

(i) $\left(\mathbb{A}_{X,01}^{\mathrm{ar}}\right)^{\perp} = \mathbb{A}_{X,01}^{\mathrm{ar}}$;

(ii) $\left(\mathbb{A}_{X,02}^{\mathrm{ar}}\right)^{\perp} = \mathbb{A}_{X,02}^{\mathrm{ar}}$;

(iii) $\left(\mathbb{A}_{X,12}^{\mathrm{ar}}(D)\right)^{\perp} = \mathbb{A}_{X,12}^{\mathrm{ar}}((\omega)-D)$.

Proof. By an abuse of notation, we will write elements of $\mathbb{A}_X^{\mathrm{fin}}$ as $(f_{C,x})_{C,x}$ or even $(f_{C,x})$, and elements of $\mathbb{A}_X^{\infty}$ as $(f_P)_P$ or even (f_P).

We begin with a proof of $\mathbb{A}_{X,01}^{\mathrm{ar}} \subset (\mathbb{A}_{X,01}^{\mathrm{ar}})^{\perp}$. Let $\mathbf{f}, \mathbf{g} \in \mathbb{A}_{X,01}^{\mathrm{ar}}$. By definition, $\mathbf{f} = (f_C)_{C,x} \times (f_P)_P$, $\mathbf{g} = (g_C)_{C,x} \times (g_P)_P$, and, for all algebraic points P of X_F, $f_{E_P} = f_P$ and $g_{E_P} = g_P$. Consequently,

$$\begin{aligned}\langle \mathbf{f}, \mathbf{g}\rangle_\omega =& \sum_{x,C} \mathrm{Res}_{C,x}(f_C g_C \omega) + \sum_P \mathrm{Res}_P(f_P g_P \omega) \\ =& \sum_C \sum_{x:x\in C} \mathrm{Res}_{C,x}(f_C g_C \omega) + \sum_P \mathrm{Res}_P(f_P g_P \omega).\end{aligned}$$

If C is a vertical curve on X, then, by the standard residue theorem for algebraic curves, see e.g. [P1], $\sum_{x:x\in C} \mathrm{Res}_{C,x}(f_C g_C \omega) = 0$. Hence

$$\begin{aligned}\langle \mathbf{f}, \mathbf{g}\rangle_\omega =& \sum_{P\in X_F} \sum_{x\in E_P} \mathrm{Res}_P(f_{E_P} g_{E_P} \omega) + \sum_P \mathrm{Res}_P(f_P g_P \omega) \\ =& \sum_{P\in X_F} \sum_{Q\in \overline{E}_P} \mathrm{Res}_Q(f_P g_P \omega).\end{aligned}$$

Here $\overline{E}_P$ denotes the Arakelov completion of E_P associated to an algebraic point $P \in X_F$, and in the last step, we have used our defining condition $f_{E_P} = f_P$ and $g_{E_P} = g_P$ for elements $\mathbf{f}, \mathbf{g}$ in $\mathbb{A}_{X,01}^{\mathrm{ar}}$. Now, by the residue theorem for $\overline{E}_P$ ([M2, Thm 5.4]), $\sum_{Q\in\overline{E}_P} \mathrm{Res}_Q(f_P g_P \omega) = 0$. Therefore, $\langle \mathbf{f}, \mathbf{g}\rangle_\omega = 0$, and $\mathbb{A}_{X,01}^{\mathrm{ar}} \subset (\mathbb{A}_{X,01}^{\mathrm{ar}})^{\perp}$.

Next, we show that $\mathbb{A}_{X,01}^{\mathrm{ar}} \supset (\mathbb{A}_{X,01}^{\mathrm{ar}})^{\perp}$, based on the following ind-pro structure on $\mathbb{A}_{X,01}^{\mathrm{ar}}$ in Corollary 14:

$$\mathbb{A}_{X,01}^{\mathrm{ar}} = \varinjlim_{D'} \varprojlim_{\substack{D' \\ D'\le D}} \mathbb{A}_{X,1}^{\mathrm{ar}}(D)\big/\mathbb{A}_{X,1}^{\mathrm{ar}}(D').$$

Let C be an irreducible curve C. Then, induced by the perfect pairing $\langle\cdot,\cdot\rangle_\omega : \mathbb{A}_X^{\mathrm{ar}} \times \mathbb{A}_X^{\mathrm{ar}} \to \mathbb{R}/\mathbb{Z}$, for any divisor D on X, by (iii), whose proof given below is independent of (i) and (ii), we obtain a pairing

$$\mathbb{A}_{X,12}^{\mathrm{ar}}(D)\big/\mathbb{A}_{X,12}^{\mathrm{ar}}(D-C) \times \mathbb{A}_{X,12}^{\mathrm{ar}}((\omega)+C-D)\big/\mathbb{A}_{X,12}^{\mathrm{ar}}((\omega)-D) \longrightarrow \mathbb{R}/\mathbb{Z}. \quad (1)$$

Moreover, directly from the definition, we have that $\mathbb{A}_{X,12}^{\mathrm{ar}}(D)\big/\mathbb{A}_{X,12}^{\mathrm{ar}}(D-C) \simeq \mathbb{A}_{C,01}^{\mathrm{ar}}$ for any divisor D. Hence, $\mathbb{A}_{X,12}^{\mathrm{ar}}((\omega)+C-D)\big/\mathbb{A}_{X,12}^{\mathrm{ar}}((\omega)-D) \simeq \mathbb{A}_{C,01}^{\mathrm{ar}}$ as well. So we can and will view (1) as a pairing on $\mathbb{A}_{C,01}^{\mathrm{ar}}$.

If C is vertical, then, there exists $\omega_C \in \Omega_{k(C)/\mathbb{F}_p}$ and an $\mathbf{a} = (a_v) \in \mathbb{A}^{\mathrm{ar}}_{C,01}$ such that (1) coincides with the pairing

$$\langle \cdot, \cdot \rangle_{\omega_C, \mathbf{a}} : \mathbb{A}^{\mathrm{ar}}_{C,01} \times \mathbb{A}^{\mathrm{ar}}_{C,01} \longrightarrow \mathbb{F}_p; \qquad (\mathbf{f}, \mathbf{g}) \mapsto \sum_v \mathrm{Res}_v(f_v g_v a_v \omega_C). \quad (2)$$

Since $\mathbb{A}^{\mathrm{ar}}_{X,01} \subset (\mathbb{A}^{\mathrm{ar}}_{X,01})^{\perp}$, and, directly from the definition, we have that

$$\mathbb{A}^{\mathrm{ar}}_{X,1}(D)/\mathbb{A}^{\mathrm{ar}}_{X,1}(D-C) \simeq \mathbb{A}^{\mathrm{ar}}_{X,1}((\omega)+C-D)/\mathbb{A}^{\mathrm{ar}}_{X,1}((\omega)-D) \simeq \mathbb{A}^{\mathrm{ar}}_{C,0} = k(C), \quad (3)$$

we conclude that $\langle \cdot, \cdot \rangle_{\omega_C, \mathbf{a}} : \mathbb{A}^{\mathrm{ar}}_{C,01} \times \mathbb{A}^{\mathrm{ar}}_{C,01} \to \mathbb{F}_p$ annihilates $k(C) \times k(C)$. But $\langle \cdot, \cdot \rangle_{\omega_C, \mathbf{a}}$ can be identified with the canonical residue pairing $\langle \cdot, \cdot \rangle : \mathbb{A}^{\mathrm{ar}}_{C,01} \times \mathbb{A}^{\mathrm{ar}}_{C,01} \to \mathbb{F}_p$ associated to C. Consequently, $K(C)^{\perp} = K(C)$.[2] In particular, with respect to the pairing $\langle ak(C), k(C) \rangle = 0$. So $\mathbf{a} \in k(C)$. Hence, if necessary, with a possible modification on ω, without loss of generality, we may and will assume $\mathbf{a} = 1$ and write $\langle \cdot, \cdot \rangle_{\omega_C, \mathbf{a}}$ simply as $\langle \cdot, \cdot \rangle_{\omega_C}$. Therefore, by (3) and the fact that $k(C)^{\perp} = k(C)$, we have

$$\Big(\mathbb{A}^{\mathrm{ar}}_{X,1}(D)/\mathbb{A}^{\mathrm{ar}}_{X,1}(D-C)\Big)^{\perp} \simeq \mathbb{A}^{\mathrm{ar}}_{X,1}((\omega)+C-D)/\mathbb{A}^{\mathrm{ar}}_{X,1}((\omega)-D).$$

Moreover, with a verbatim change, the same discussion is valid for horizontal curves as well. Consequently, by applying this repeatedly, we have, for any irreducible curves C_1, C_2, the following commutating diagram with exact rows

$$\begin{array}{ccccc} A(D-C_1)/A(D-C_1-C_2) & \hookrightarrow & A(D)/A(D-C_1-C_2) & \twoheadrightarrow & A(D)/A(D-C_1) \\ \| & & \downarrow & & \| \\ G\cap B(D-C_1)/G\cap B(D-C_1-C_2) & \hookrightarrow & G\cap B(D)/G\cap B(D-C_1-C_2) & \twoheadrightarrow & G\cap B(D)/G\cap B(D-C_1) \end{array}$$

where, to save space, we set $A := \mathbb{A}^{\mathrm{ar}}_{X,1}$, $B := \mathbb{A}^{\mathrm{ar}}_{X,12}$ and $G := \big(\mathbb{A}^{\mathrm{ar}}_{X,01}\big)^{\perp}$. Consequently, the vertical map in the middle is surjective. On the other hand, since $\mathbb{A}^{\mathrm{ar}}_{X,01} \subset \big(\mathbb{A}^{\mathrm{ar}}_{X,01}\big)^{\perp}$, this same map is also injective. Therefore, for any $D' \le D$,

$$\Big(\mathbb{A}^{\mathrm{ar}}_{X,1}(D)/\mathbb{A}^{\mathrm{ar}}_{X,1}(D')\Big)^{\perp} \simeq \mathbb{A}^{\mathrm{ar}}_{X,1}((\omega)-D')/\mathbb{A}^{\mathrm{ar}}_{X,1}((\omega)-D).$$

with respect to our pairing

$$\mathbb{A}^{\mathrm{ar}}_{X,1}(D)/\mathbb{A}^{\mathrm{ar}}_{X,1}(D') \times \mathbb{A}^{\mathrm{ar}}_{X,1}((\omega)-D')/\mathbb{A}^{\mathrm{ar}}_{X,1}((\omega)-D) \longrightarrow \mathbb{R}/\mathbb{Z}.$$

Consequently, we have

$$\big(\mathbb{A}^{\mathrm{ar}}_{X,01}\big)^{\perp} = \mathbb{A}^{\mathrm{ar}}_{X,01},$$

since, by (iii) again,

$$\mathbb{A}^{\mathrm{ar}}_{X,01} = \lim_{\longrightarrow D'} \lim_{\substack{\longleftarrow D' \\ D' \le D}} \mathbb{A}^{\mathrm{ar}}_{X,1}(D)/\mathbb{A}^{\mathrm{ar}}_{X,1}(D') = \lim_{\longleftarrow D} \lim_{\substack{\longrightarrow D' \\ D' \le D}} \mathbb{A}^{\mathrm{ar}}_{X,1}((\omega)-D')/\mathbb{A}^{\mathrm{ar}}_{X,1}((\omega)-D).$$

This proves (i).

[2] It is well-known that, see e.g., [Iw, §4], if χ is a non-zero character on $\mathbb{A}^{\mathrm{ar}}_{C,01}$ such that $\chi(k(C)) = \{0\}$, then the induced pairing $\langle \cdot, \cdot \rangle_{\chi} : \mathbb{A}^{\mathrm{ar}}_{C,01} \times \mathbb{A}^{\mathrm{ar}}_{C,01} \to \mathbb{F}_q$; $(\mathbf{f}, \mathbf{g}) \mapsto \chi(\mathbf{f} \cdot \mathbf{g})$ is perfect and $k(C)^{\perp} = k(C)$.

To prove (ii), we start with the inclusion $\mathbb{A}^{\rm ar}_{X,02} \subset (\mathbb{A}^{\rm ar}_{X,02})^{\perp}$. By definition, every element $\mathbf{f} \in \mathbb{A}^{\rm ar}_{X,02}$ can be written as $\mathbf{f} = (f_x)_{C,x} \times (f_P)_P$ with $f_{C,x} = f_x$ and $(f_P) = (f)$ for some $f \in k(X_F)$ (since, by definition, the 02 type adeles are independent of one dimensional curves). Thus, for $\mathbf{f}, \mathbf{g} \in \mathbb{A}^{\rm ar}_{X,02}$, we have

$$\begin{aligned}\langle \mathbf{f}, \mathbf{g} \rangle =& \sum_{C,x} \mathrm{Res}_{C,x}(f_{C,x} g_{C,x} \omega) + \sum_P \mathrm{Res}_P(f_P g_P \omega) \\ =& \sum_x \sum_{C: C \ni x} \mathrm{Res}_{C,x}(f_x g_x \omega) + \sum_P \mathrm{Res}_P(f g \omega).\end{aligned}$$

Note that for a fixed x, f_x and g_x are fixed. Thus, by the residue theorem for the point x ([M1, Thm 4.1]), we have $\sum_{C:C\ni x} \mathrm{Res}_{C,x}(f_x g_x \omega) = 0$. So

$$\langle \mathbf{f}, \mathbf{g} \rangle = \sum_x 0 + \sum_P \mathrm{Res}_P(f g \omega) = \sum_P \mathrm{Res}_P(f g \omega).$$

On the other hand, since f, $g \in k(X_F)$ and ω is a rational differential on X_F, the standard residue formula for the curve X_F/F (see e.g., [S, §II.7, Prop. 6]) implies that $\sum_P \mathrm{Res}_P(f g \omega) = 0$. Hence

$$\langle \mathbf{f}, \mathbf{g} \rangle = 0, \qquad \forall\ \mathbf{f}, \mathbf{g} \in \mathbb{A}^{\rm ar}_{X,02}.$$

Therefore, $\mathbb{A}^{\rm ar}_{X,02} \subset (\mathbb{A}^{\rm ar}_{X,02})^{\perp}$.

For the opposite direction $\mathbb{A}^{\rm ar}_{X,02} \supset (\mathbb{A}^{\rm ar}_{X,02})^{\perp}$, similarly as in (i), we, in theory, can use the following ind-pro structure on $\mathbb{A}^{\rm ar}_{X,02}$:

$$\mathbb{A}^{\rm ar}_{X,02} = \lim_{\longrightarrow D'} \lim_{\substack{\longleftarrow D' \\ D' \leq D}} \mathbb{A}^{\rm ar}_{X,2}(D) / \mathbb{A}^{\rm ar}_{X,2}(D').$$

However, due to the lack of details for horizontal differential theory in literature, we decide to first use this ind-pro structure to merely prove the part that if $\mathbf{f} = (f_{C,x})_{C,x} \times (f_P)_P \in (\mathbb{A}^{\rm ar}_{X,02})^{\perp}$, for vertical C's, $f_{C,x} = f_x$; and then to take a more classical approach for the rest.

Choose $\mathbf{f} = (f_{C,x})_{C,x} \times (f_P)_P \in (\mathbb{A}^{\rm ar}_{X,02})^{\perp}$. Then by our assumption, for any element $\mathbf{g} = (g_x)_{C,x} \times (g)_P \in \mathbb{A}^{\rm ar}_{X,02}$, we have

$$0 = \langle \mathbf{f}, \mathbf{g} \rangle = \sum_x \sum_{C: C \ni x} \mathrm{Res}_{C,x}(f_{C,x} g_x \omega) + \sum_P \mathrm{Res}_P(f_P g \omega). \tag{4}$$

Now note that the element $(g_x)_{C,x} \in \mathbb{A}^{\rm ar}_{X,02}$ and the element $g \in k(X_F)$ can be changed totally independently, we conclude that both of the summations, $\sum_x \sum_{C:C\ni x} \mathrm{Res}_{C,x}(f_{C,x} g_x \omega)$ and $\sum_P \mathrm{Res}_P(f_P g \omega)$, are constants independent of $\mathbf{g}$. This then implies that both of them are 0. Indeed, since $\sum_P \mathrm{Res}_P(f_P g \omega)$ is a constant independent of g, by choosing g to be constant function, we conclude that $\sum_P \mathrm{Res}_P(f_P g \omega) \equiv 0,\ \ \forall g \in k(X_K)$. Consequently, by (4), we have, from (2),

$$\sum_x \sum_{C: C \ni x} \mathrm{Res}_{C,x}(f_{C,x} g_x \omega) \equiv 0, \ \ \forall (g_x) \in A^{\rm ar}_{X,02}. \tag{5}$$

To end the proof, we need to show that $f_{C,x}$'s are independent of C and f_P are independent of P. First, we treat the case when C is vertical. As said above, we will use the associated ind-pro structures. So, assume for now that C is vertical. Then, for any divisor D on X, we have

$$\mathbb{A}^{\mathrm{ar}}_{X,2}(D)\big/\mathbb{A}^{\mathrm{ar}}_{X,2}(D-C) \simeq \mathbb{A}_{C,1}(D|_C),$$
$$\mathbb{A}^{\mathrm{ar}}_{X,2}((\omega)+C-D)\big/\mathbb{A}^{\mathrm{ar}}_{X,2}((\omega)+C-(D-C)) \simeq \mathbb{A}_{C,1}((\omega'_C)-D|_C),$$

for a certain $\omega'_C \in \Omega_{k(C)/\mathbb{F}_p}$ satisfying $(\omega'_C) = \big((\omega)+C\big)|_C$ by the adjunction formula. We claim that $(\omega'_C) = (\omega_C)$. Indeed, since $\mathbb{A}_{C,1}(D|_C)^\perp = \mathbb{A}_{C,1}((\omega_C)-D|_C)$ and $\mathbb{A}^{\mathrm{ar}}_{X,02} \subset \big(\mathbb{A}^{\mathrm{ar}}_{X,02}\big)^\perp$, $\mathbb{A}_{C,1}((\omega'_C)-D|_C) \subset \mathbb{A}_{C,1}((\omega_C)-D|_C)$. This implies that $(\omega_C) \geq (\omega'_C)$ and hence $(\omega_C) = (\omega'_C)$, because there is no $f \in k(C)$ such that $(f) > 0$. Thus, with respect to the canonical residue pairing on C, we have $\mathbb{A}_{C,1}(D|_C)^\perp = \mathbb{A}_{C,1}((\omega'_C)-D|_C)$. Consequently, as in (i), from

$$\mathbb{A}^{\mathrm{ar}}_{X,02} = \lim_{\longrightarrow D'} \lim_{\substack{\longleftarrow D' \\ D' \leq D}} \mathbb{A}^{\mathrm{ar}}_{X,2}(D)\big/\mathbb{A}^{\mathrm{ar}}_{X,2}(D') = \lim_{\longleftarrow D} \lim_{\substack{\longrightarrow D' \\ D' \leq D}} \mathbb{A}^{\mathrm{ar}}_{X,2}((\omega)-D')\big/\mathbb{A}^{\mathrm{ar}}_{X,2}((\omega)-D),$$

we conclude that $f_{C,x} = f_x \in k(X)_x$ and hence independent of C.

To prove the rest, we take a classical approach with a use of Chinese reminder theorem, using an idea in the proof of Proposition 1 of [P1]. To be more precise, fix $x_0 \in X$ and a prime divisor H on X satisfying that $x_0 \in H$ and that $V := \operatorname{Spec} \mathcal{O}_{X,x_0} - H$ is affine. We claim that for any family of prime divisors D_j on V such that $D_i \cap D_j = \emptyset$ if $i \neq j$, and any rational functions $f_0, f_1, \ldots, f_n$ on V, and any fixed divisor D supported on D_i's, there exists a rational function g such that

$$\begin{cases} \mathrm{ord}_{D_i}(f_i - g) \geq \mathrm{ord}_{D_i}(D), & i = 0, 1, \ldots, n, \\ \mathrm{ord}_{D_i}(g) \geq \mathrm{ord}_{D_i}(D), & i \notin \{0, 1, \ldots, n\} \end{cases}$$

Indeed, by clearing the common denominators for f_i's, (with a modification of f_i's if necessary,) we may assume that f_i's are all regular. Then by applying the Chinese reminder theorem to the fractional ideals $\mathfrak{p}_i^{\mathrm{ord}_{D_i}(D)}, i = 0, 1, \ldots, n$, and $\cap_{i \notin \{0,1,\ldots,n\}} \mathfrak{p}_i^{\mathrm{ord}_{D_i}(D)}$, where $\mathfrak{p}_i$ are the prime ideas associated to the prime divisors D_i, we see the existence of such a g.

Associated to $\mathbf{f} \in \Big(\mathbb{A}^{\mathrm{ar}}_{X,02}\Big)^\perp$, form a new adele $\mathbf{f}' \in \mathbb{A}^{\mathrm{ar}}_{X,012}$ by setting $f'_{C,x} = f_{C,x} - f_{H,x}$ where H is a fixed vertical curve. Since, as proved above, for vertical H, $f_{H,x} = f_x \in k(X)_x$, this is well defined. Then

$$\begin{aligned} &\sum_{C:C \ni x_0} \mathrm{Res}_{C,x_0}(f'_{C,x_0} g_{x_0} \omega) \\ =& \sum_{C:C \ni x_0} \mathrm{Res}_{C,x_0}(f_{C,x_0} g_{x_0} \omega) - \sum_{C:C \ni x_0} \mathrm{Res}_{C,x_0}(f_{H,x_0} g_{x_0} \omega). \end{aligned}$$

The first sum is zero, since we have (5) by our choice of $\mathbf{f}$. The second sum, being taken over all prime curves passing through x_0, is zero as well, since we can apply the residue theorem for the point x_0 as above (with f_{H,x_0} being independent of C). That is to say,

$$\sum_{C:C \ni x_0} \mathrm{Res}_{C,x_0}(f'_{C,x_0} g_{x_0} \omega) = 0.$$

Now, applying the above existence to obtain a g satisfying that for any fixed rational function f_0 and any fixed curve $C_0 \ni x_0$, we have

$$\begin{cases} \mathrm{ord}_C(f'_{C,x_0}) + \mathrm{ord}_C(f_0 - g) + \mathrm{ord}_C(\omega) \geq 0, & C = C_0 \\ \mathrm{ord}_C(f'_{C,x_0}) + \mathrm{ord}_C(g) + \mathrm{ord}_C(\omega) \geq 0, & C \neq C_0, H. \end{cases}$$

Consequently, by the definition of the residue map, with $f'_{H,x_0} \equiv 0$ in mind, we get, for any $f_0 \in k(X)$ and the corresponding g just chosen,

$$\begin{aligned} 0 = \sum_{C:C\ni x_0} \mathrm{Res}_{C,x_0}(f'_{C,x_0} g\omega) &= \sum_{C:C\ni x_0, \{C:C\neq C_0,H\}\cup\{C:C=C_0,H\}} \mathrm{Res}_{C,x_0}(f'_{C,x_0} g\omega) \\ = \sum_{C:C\ni x_0, C=C_0,H} \mathrm{Res}_{C,x_0}(f'_{C,x_0} g\omega) &= \mathrm{Res}_{C_0,x_0}(f'_{C_0,x_0} g\omega) = \mathrm{Res}_{C_0,x_0}(f'_{C_0,x_0} f_0\omega). \end{aligned}$$

Since the last quantity is always zero for all f_0, this then implies that $f'_{C,x_0} = 0$, namely, $f_{C_0,x_0} = f_{H,x_0}$.

To end the proof of (ii), we still need to show that $f_P = f_{P_0}$ for a fixed $P_0 \in X_K$ and all $P \in X_K$. But this is amount to a use of a similar argument just said again, based on Chinese reminder theorem. See e.g., [Iw, §4]. We leave the details for the reader. Thus, if $\mathbf{f} = (f_{C,x})_{C,x} \times (f_P)_P \in (\mathbb{A}^{\mathrm{ar}}_{X,02})^\perp$, then $f_{C,x} = f_{C_0,x}$ and $f_P = f_{P_0}$ for fixed C_0 and P_0. Therefore, $\mathbf{f} \in \mathbb{A}^{\mathrm{ar}}_{X,02}$. That is to say, $(\mathbb{A}^{\mathrm{ar}}_{X,02})^\perp = \mathbb{A}^{\mathrm{ar}}_{X,02}$. This proves (ii).

Finally, we prove (iii). The inclusion $A^{\mathrm{ar}}_{X,12}((\omega) - D) \subset (A^{\mathrm{ar}}_{X,12}(D))^\perp$ is easy. Indeed, for $\mathbf{f} = (f_{C,x}) \times (f_P) \in A^{\mathrm{ar}}_{X,12}((\omega) - D)$, $\mathbf{g} = (g_{C,x}) \times (g_P) \in A^{\mathrm{ar}}_{X,12}(D)$, we have $\langle \mathbf{f}, \mathbf{g} \rangle = \sum_{C,x} \mathrm{Res}_{C,x}(f_{C,x} g_{C,x}\omega) + \sum_P \mathrm{Res}_P(f_P g_P \omega) = 0$, since every term in each of these two summations is zero by definition.

To prove the other direction $A^{\mathrm{ar}}_{X,12}((\omega) - D) \supset (A^{\mathrm{ar}}_{X,12}(D))^\perp$, we make the following preparations. Set $\widetilde{\pi} : X \to \mathrm{Spec}\,\mathcal{O}_F \to \mathrm{Spec}\,\mathbb{Z}$. Then, by [M1, Thm 5.7], the dualizing sheaf $\omega_{\widetilde{\pi}}$ of $\widetilde{\pi}$ is given by, for an open subset $U \subseteq X$,

$$\omega_{\widetilde{\pi}}(U) = \big\{\omega \in \Omega_{k(X)/\mathbb{Q}} \mid \mathrm{Res}_{C,x}(f\omega) = 0 \quad \forall x \in C(\subset U),\ \forall f \in \mathcal{O}_{X,C}\big\}.$$

By a similar argument as in [M1] (used to prove the above result), we have, for a fixed curve C_0,

$$\omega_{\widetilde{\pi},C_0} = \big\{\omega \in \Omega_{k(X)/\mathbb{Q}} \mid \mathrm{Res}_{C_0,x}(f\omega) = 0 \quad \forall\, x \in C_0,\ f \in \mathcal{O}_{X,C_0}\big\}.$$

This is nothing but the collection of differentials ω satisfying $\mathrm{ord}_{C_0}((\omega)) \geq 0$. Moreover, we have, for a fixed pair $x_0 \in C_0$,

$$\omega_{\widetilde{\pi},C_0} \otimes_{\mathcal{O}_{C_0}} \mathcal{O}_{C_0,x_0} = \big\{\omega \in \Omega^{\mathrm{cts}}_{k(X)_{C_0,x_0}/\mathbb{Q}_{\widetilde{\pi}(x_0)}} \mid \mathrm{Res}_{C_0,x_0}(f\omega) = 0 \quad \forall\, f \in \mathcal{O}_{C_0,x_0}\big\}.$$

This is simply the collection of differentials ω satisfying $\mathrm{ord}_{C_0}((\omega)) \geq 0$. From these, we conclude that the following conditions are equivalent for a non-zero differential $\omega \in \Omega_{k(X)/\mathbb{Q}}$ and a fixed pair $x_0 \in C_0$;

$(1)_C$ $\forall f \in \mathcal{O}_{X,C_0}$ $\mathrm{Res}_{\mathrm{C_0,x_0}}(f\omega) = 0$.

$(2)_C$ $\mathrm{ord}_{C_0}((\omega)) \geq 0$.

Furthermore, a similar argument for points instead of curves also gives us that the following conditions are equivalent for a fixed point P;

$(1)_P$ $\forall f \in \mathcal{O}_{X_F,P_0}$ $\mathrm{Res}_{P_0}(f\omega) = 0$.
$(2)_P$ $\mathrm{ord}_{P_0}((\omega)) \geq 0$.

Now we are ready to continue our proof. Let $\mathbf{f} = (f_{C,x}) \times (f_P) \in A^{\mathrm{ar}}_{X,12}(D)^\perp$ and $\mathbf{g} = (g_{C,x}) \times (g_P) \in A^{\mathrm{ar}}_{X,12}(D)$. Then

$$0 = \langle \mathbf{f}, \mathbf{g} \rangle = \sum\nolimits_{C,x} \mathrm{Res}_{C,x}(f_{C,x}g_{C,x}\omega) + \sum\nolimits_{P} \mathrm{Res}_P(f_P g_P \omega).$$

Thus, by the independence of components of adeles, we see that, for fixed (C,x) and P, $\mathrm{Res}_{C,x}(f_{C,x}g_{C,x}\omega) = 0$ and $\mathrm{Res}_P(f_P g_P \omega) = 0$. This, together with the equivalence between $(1)_C$ and $(2)_C$ and the equivalence between $(1)_P$ and $(2)_P$, we conclude that $\mathbf{f} = (f_{C,x}) \times (f_P) \in A^{\mathrm{ar}}_{X,12}((\omega) - D)$, and hence complete the proof.

2.4 Arithmetic Cohomology Groups

2.4.1 Definitions

Let X be an arithmetic surface and D be a Weil divisor on X. By Lemma 13, particularly, with (iv), we obtain an arithmetic adelic complex. Accordingly, we define the *arithmetic cohomology groups* $H^i_{\mathrm{ar}}(X,D)$ by $H^i(\mathbb{A}^*_{\mathrm{ar}}(X,D), d^*)$, the *i-th* cohomology groups of the complex $(\mathbb{A}^*_{\mathrm{ar}}(X,D), d^*)$, $i = 0,1,2$.

Proposition 16. *(i) The arithmetic cohomology groups $H^i_{\mathrm{ar}}(X,D)$ of D on X, $i = 0,1,2$, are given by*

$$\begin{aligned}
&H^0_{\mathrm{ar}}(X,D) = \mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,02} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D);\\
&H^1_{\mathrm{ar}}(X,D) =\\
&= \Big((\mathbb{A}^{\mathrm{ar}}_{X,01} + \mathbb{A}^{\mathrm{ar}}_{X,02}) \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D) \Big) \Big/ \Big(\mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D) + \mathbb{A}^{\mathrm{ar}}_{X,02} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D) \Big);\\
&H^2_{\mathrm{ar}}(X,D) = \mathbb{A}^{\mathrm{ar}}_{X,012} \Big/ \Big(\mathbb{A}^{\mathrm{ar}}_{X,01} + \mathbb{A}^{\mathrm{ar}}_{X,02} + \mathbb{A}^{\mathrm{ar}}_{X,12}(D) \Big).
\end{aligned}$$

(ii) There exist natural isomorphisms

$$\begin{aligned}
&H^1_{\mathrm{ar}}(X,D)\\
\simeq &\Big(\mathbb{A}^{\mathrm{ar}}_{X,01} \cap \big(\mathbb{A}^{\mathrm{ar}}_{X,02} + \mathbb{A}^{\mathrm{ar}}_{X,12}(D) \big) \Big) \Big/ \Big(\mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,02} + \mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D) \Big)\\
\simeq &\Big(\mathbb{A}^{\mathrm{ar}}_{X,02} \cap \big(\mathbb{A}^{\mathrm{ar}}_{X,01} + \mathbb{A}^{\mathrm{ar}}_{X,12}(D) \big) \Big) \Big/ \Big(\mathbb{A}^{\mathrm{ar}}_{X,01} \cap \mathbb{A}^{\mathrm{ar}}_{X,02} + \mathbb{A}^{\mathrm{ar}}_{X,02} \cap \mathbb{A}^{\mathrm{ar}}_{X,12}(D) \Big)
\end{aligned}$$

Proof. This is an arithmetic analogue of [P1]. Indeed, (i) comes directly from the definition, and via the first and second isomorphism theorems in elementary group theory, (ii) is obtained using a direct group theoretic calculation for our arithmetic adelic complex.

As mentioned above, the cohomology groups for the divisor D defined here coincide with the adelic global cohomology defined in §1.2.3 associated to the invertible sheaf $\mathcal{O}_X(D)$. That is, $H^i_{\mathrm{ar}}(X,D) = H^i_{\mathrm{ar}}(X, \mathcal{O}_X(D))$. In fact, our general construction in §1.2.3 for quasi-coherent sheaves over higher dimensional arithmetic varieties is obtained modeling our constructions for arithmetic curves and arithmetic surfaces.

2.4.2 Inductive long exact sequences

(V) Vertical Curves

Just like the classical cohomology theory, arithmetic cohomology groups also admit an inductive structure related to vertical geometric curves on arithmetic surfaces. More precisely, we have the following

Proposition 17. *Let C be an irreducible vertical curve of X, then, for any D, we have the long exact sequence of cohomology groups*

$$\begin{aligned} 0 \to & H^0_{\rm ar}(X,C) \to H^0_{\rm ar}(X,D+C) \to H^0_{\rm ar}(C,(D+C)|_C) \\ & \to H^1_{\rm ar}(X,D) \to H^1_{\rm ar}(X,D+C) \to H^1_{\rm ar}(C,(D+C)|_C) \\ & \to H^2_{\rm ar}(X,D) \to H^2_{\rm ar}(X,D+C) \to 0. \end{aligned}$$

Here $H^i_{\rm ar}(C,(D+C)|_C)$, $i=0,1$ are the usual cohomology groups for vertical geometric curves.

Proof. For C be an irreducible vertical curve in X. Then from definition, one calculates that

(a) $\mathbb{A}^{\rm ar}_{X,12}(D+C)/A^{\rm ar}_{X,12}(D) = \mathbb{A}^{\rm fin}_{X,12}(D+C)/A^{\rm fin}_{X,12}(D) \oplus \{0\}\mathbb{A}_{C,01} \oplus \{0\}$.

(b) $\mathbb{A}^{\rm ar}_{X,1}(D+C)/\mathbb{A}^{\rm ar}_{X,1}(D) = k(C)$ as there are neither changes along horizontal curves nor along X_F.

(c) $\mathbb{A}^{\rm ar}_{X,2}(D+C)/\mathbb{A}^{\rm ar}_{X,2}(D)$

$$= \mathbb{A}^{\rm fin}_{X,2}(D_{\rm fin}+C)/\mathbb{A}^{\rm fin}_{X,2}(D_{\rm fin}) \oplus \{0\} = \mathbb{A}_C\Big((D_{\rm fin}+C)|_C\Big) \oplus \{0\}.$$

Consequently, for the morphism

$$\begin{array}{ccc} \mathbb{A}^{\rm ar}_{X,1}(D+C)/\mathbb{A}^{\rm ar}_{X,1}(D) \oplus \mathbb{A}^{\rm ar}_{X,2}(D+C)/\mathbb{A}^{\rm ar}_{X,2}(D) & \xrightarrow{\phi} & \mathbb{A}^{\rm ar}_{X,12}(D+C)/A^{\rm ar}_{X,12}(D), \\ (x,y) & \mapsto & x-y, \end{array}$$

we conclude that

(d) $\operatorname{Ker}\phi$ is given by

$$\Big(\mathbb{A}^{\rm ar}_{X,1}(D+C)/\mathbb{A}^{\rm ar}_{X,1}(D)\Big) \cap \Big(\mathbb{A}^{\rm ar}_{X,2}(D+C)/\mathbb{A}^{\rm ar}_{X,2}(D)\Big) = H^0_{\rm ar}(C,(D+C)|_C),$$

(e) $\operatorname{Coker}\phi$ is given by

$$\begin{aligned} &\Big(\mathbb{A}^{\rm ar}_{X,12}(D+C)/A^{\rm ar}_{X,12}(D)\Big)\Big/\Big(\mathbb{A}^{\rm ar}_{X,12}(D+C)/A^{\rm ar}_{X,12}(D) + \mathbb{A}^{\rm ar}_{X,12}(D+C)/A^{\rm ar}_{X,12}(D)\Big) \\ &\quad = H^1_{\rm ar}(C,(D+C)|_C). \end{aligned}$$

Therefore, by definition, we have the long exact sequence

$$\begin{aligned} 0 \to & H^0_{\rm ar}(X,C) \to H^0_{\rm ar}(X,D+C) \to H^0_{\rm ar}(C,(D+C)|_C) \\ & \to H^1_{\rm ar}(X,D) \to H^1_{\rm ar}(X,D+C) \to H^1_{\rm ar}(C,(D+C)|_C) \\ & \to H^2_{\rm ar}(X,D) \to H^2_{\rm ar}(X,D+C) \to 0. \end{aligned}$$

This then completes the proof.

(H) Horizontal Curves

For horizontal curve E_P corresponding to an algebraic point P of X_F, we have

(a) $\mathbb{A}^{\rm ar}_{X,12}(D+E_P)/A^{\rm ar}_{X,12}(D)$

$$\begin{aligned}&=\mathbb{A}^{\rm fin}_{X,12}(D+E_P)/A^{\rm fin}_{X,12}(D)\oplus\mathbb{A}_{X_F}(D_F+P)/A_{X_F}(D_F)\otimes_{\mathbb{Q}}\mathbb{R}\\&=\mathbb{A}_{E_P,01}\oplus k(E_P)\otimes_{\mathbb{Q}}\mathbb{R}=\mathbb{A}^{\rm ar}_{E_P};\end{aligned}$$

(b) $\mathbb{A}^{\rm ar}_{X,1}(D+E_P)/\mathbb{A}^{\rm ar}_{X,1}(D)=k(E_P)$, diagonally embedded in

$$\mathbb{A}^{\rm fin}_{X,1}(D_{\rm fin}+E_P)/\mathbb{A}^{\rm fin}_{X,1}(D_{\rm fin})\oplus\mathbb{A}_{X_F}(D_F+P)/\mathbb{A}_{X_F}(D_F)\otimes_{\mathbb{Q}}\mathbb{R};$$

(c) $\mathbb{A}^{\rm ar}_{X,2}(D+E_P)/\mathbb{A}^{\rm ar}_{X,2}(D)$

$$=\mathbb{A}_{E_P}\Big((D_{\rm fin}+E_P)|_{E_P}\Big)\oplus H^0(X_F,D_F+E_P)\otimes\mathbb{R}/H^0(X_F,D_F)\otimes\mathbb{R}.$$

Similarly, we have the corresponding morphism

$$\begin{array}{ccc}\mathbb{A}^{\rm ar}_{X,1}(D+E_P)/\mathbb{A}^{\rm ar}_{X,1}(D)\oplus\mathbb{A}^{\rm ar}_{X,2}(D+E_P)/\mathbb{A}^{\rm ar}_{X,2}(D) & \xrightarrow{\varphi} & \mathbb{A}^{\rm ar}_{X,12}(D+E_P)/A^{\rm ar}_{X,12}(D)\\(x,y) & \mapsto & x-y,\end{array}$$

and hence obtain the following proposition in parallel.

Proposition 18. *Let E_P be the horizontal curve E_P corresponding to an algebraic point P of X_F, then, for any D, We have the long exact sequence*

$$\begin{aligned}0\to&H^0_{\rm ar}(X,E_P)\to H^0_{\rm ar}(X,D+E_P)\to{\rm Ker}\,\varphi\\&\to H^1_{\rm ar}(X,D)\to H^1_{\rm ar}(X,D+E_P)\to{\rm Coker}\,\varphi\\&\to H^2_{\rm ar}(X,D)\to H^2_{\rm ar}(X,D+E_P)\to 0.\end{aligned}$$

However, unlike for vertical curves, we do not have the group isomorphisms between ${\rm Ker}\,\varphi$, resp. ${\rm Coker}\,\varphi$, and $H^0_{\rm ar}(E_P,(D+E_P)|_{E_P})$, resp. $H^1_{\rm ar}(E_P,(D+E_P)|_{E_P})$. This is in fact not surprising: different from vertical curves, for the arithmetic cohomology, there is no simple additive law with respect to horizontal curves when count these arithmetic groups: In Arakelov theory, we only have

$$\chi_{\rm ar}(X,D+E_P)=\chi_{\rm ar}(X,D)+\chi_{\rm ar}(E_P,(D+E_P)|_{E_P})-\frac{1}{2}d_\lambda(E)\qquad(6)$$

with discrepancy $-\frac{1}{2}d_\lambda(E)$ resulting from Green's functions. (See e.g., [L, p.114].)

On the other hand, recall that, on generic fiber X_F, we have the long exact sequence of cohomology groups

$$\begin{aligned}0\to&H^0(X_F,D_F)\to H^0(X_F,D_F+P)\to\mathcal{O}_{X_F}(D+P)|_P\to Q\to 0\\&\text{(with Q defined by}\qquad 0\to Q\to H^1(X_F,D_F)\to H^1(X_F,D_F+P)\to 0),\end{aligned}\qquad(7)$$

and that, in Arakelov theory, see e.g., [L, VI, particularly, p.140], what really used is the much rough version $\lambda(D_F+P)\simeq\lambda(D_F)\otimes\mathcal{O}_{X_F}(D+P)|_P$ where

λ denotes the Grothendieck-Mumford determinant. One checks that the exact sequence (6) does appear in our calculation above. Indeed, for curve X_F/F,

$$H^0(X_F, D_F) = k(X_F) \cap \mathbb{A}_{X_F}(D_F) \quad \& \quad H^1(X_F, D_F) = \mathbb{A}_{X_F}/k(X_F) + \mathbb{A}_{X_F}(D_F).$$

Consequently, (7) is equivalent to the exact sequence

$$0 \to H^0(X_F, D_F + P)/H^0(X_F, D_F) \to \mathbb{A}_{X_F}(D_F + P)/A_{X_F}(D_F) \to Q \to 0$$
$$0 \to Q \to H^1(X_F, D_F) \to H^1(X_F, D_F + P) \to 0.$$

(Note that $\mathbb{A}_{X_F}(D_F + P)/A_{X_F}(D_F)$ is supported only on P.) Clearly, all this can be read from the calculations in (a,b,c) and the morphism φ above. So our construction offers a much more refined structure topologically.

2.4.3 Duality of cohomology groups

Let $\pi : X \to \operatorname{Spec} \mathcal{O}_F$ be an arithmetic surface defined over the ring of integers of a number field F. Then we have the adelic space $\mathbb{A}_X^{\mathrm{ar}}$, its level two subspaces $\mathbb{A}_{01}^{\mathrm{ar}}$, $\mathbb{A}_{02}^{\mathrm{ar}}$ and $\mathbb{A}_{12}^{\mathrm{ar}}(D)$, and hence the cohomology groups $H^i_{\mathrm{ar}}(X, \mathcal{O}_X(D))$ associated to a Weil divisor D on X. Moreover, there is a natural ind-pro structure on $\mathbb{A}_X^{\mathrm{ar}}$

$$\mathbb{A}_X^{\mathrm{ar}} = \lim_{\longrightarrow D} \lim_{\longleftarrow E: E \leq D} \mathbb{A}_{X,12}(D) \big/ \mathbb{A}_{X,12}(E).$$

Note that $\mathbb{A}_{X,12}(D) \big/ \mathbb{A}_{X,12}(E)$'s are locally compact topological spaces. Consequently, as explained in §3, the next section, induced from the projective limit, we get a natural final topology on $\mathbb{A}_{X,12}(D) = \lim_{\longleftarrow E: E \leq D} \mathbb{A}_{X,12}(D) \big/ \mathbb{A}_{X,12}(E)$; similarly, induced from the inductive limit, we get a natural initial topology on $\mathbb{A}_X^{\mathrm{ar}} = \lim_{\longrightarrow D} \mathbb{A}_{X,12}(D)$. Moreover, by Theorem II, we know that all three level two subspaces $\mathbb{A}_{01}^{\mathrm{ar}}$, $\mathbb{A}_{02}^{\mathrm{ar}}$ and $\mathbb{A}_{12}^{\mathrm{ar}}(D)$ are closed in $\mathbb{A}_X^{\mathrm{ar}}$. Consequently, we obtain natural topological structures on arithmetic cohomology groups $H^i_{\mathrm{ar}}(X, \mathcal{O}_X(D))$ induced from the canonical topology of $\mathbb{A}_X^{\mathrm{ar}}$. Our main theorem here is the following

Theorem 19. *Let X be an arithmetic surface and D be a Weil divisor on X. Then as topological groups, we have natural isomorphisms*

$$\widehat{H^i_{\mathrm{ar}}(X, D)} \simeq H^{2-i}_{\mathrm{ar}}((\omega) - D) \qquad i = 0, 1, 2.$$

Recall that, for a topology space T, its topological dual is defined by $\widehat{T} := \{\phi : T \to \mathbb{S}^1 \text{ continuous}\}$ together with an compact-open topology. (See e.g., §3.1.1 for details.)

This theorem is proved at the end of this paper, after we expose some basic structural results for the ind-pro topology on $\mathbb{A}_X^{\mathrm{ar}}$ in the next section.

3 Ind-Pro Topology in Dimension Two

In this section, we establish some basic properties for ind-pro topologies on various adelic spaces associated to arithmetic surfaces. This may be viewed as a natural generalization of a well-known topological theory for one dimensional adeles (see e.g. [Iw], [T]). Our main result here is the following

Theorem II. *Let X be an arithmetic surface. Then with respect to the canonical ind-pro topology on $\mathbb{A}_X^{\mathrm{ar}}$, we have*
(1) Level two subspaces $\mathbb{A}_{X,01}^{\mathrm{ar}}, \mathbb{A}_{X,02}^{\mathrm{ar}}$ and $\mathbb{A}_{X,12}^{\mathrm{ar}}(D)$ are closed in $\mathbb{A}_X^{\mathrm{ar}}$;
(2) $\mathbb{A}_X^{\mathrm{ar}}$ is a Hausdorff, complete, and compact oriented topological group;
(3) $\mathbb{A}_X^{\mathrm{ar}}$ is self-dual. That is, as topological groups,

$$\widehat{\mathbb{A}_X^{\mathrm{ar}}} \simeq \mathbb{A}_X^{\mathrm{ar}}.$$

3.1 Ind-pro topologies on adelic spaces

3.1.1 Ind-pro topological spaces and their duals

To begin with, let us recall some basic topological constructions for inductive limits and projective limits of topological spaces.
(1) Let $\{G_m\}_m$ be an inductive system of topological spaces, $G := \lim\limits_{\longrightarrow m} G_m$ with structure maps $\iota_m : G_m \to G$. Then, the inductive topology on G is defined by assigning subsets U of G to be open, if $\iota_m^{-1}(U)$ is open in G_m for each m. Inductive topology is also called the final topology since it is the finest topology on G such that $\iota_m : G_m \to G$ are continuous.
(2) Let $\{G_n\}_n$ be a projective system of topological spaces, $G := \lim\limits_{\longleftarrow n} G_n$ with structure maps $\pi_n : G \to G_n$. Then, the projective topology on G is defined as the one generated by open subsets $\pi_n^{-1}(U_n)$, where U_n are open subsets of G_n. Projective topology is also called the initial topology since it is the coarsest topology on G such that $\pi_n : G \to G_n$ are continuous.

For a topological space T, denote by $\widehat{T} := \{f : T \to \mathbb{S}^1 \text{ continuous}\}$. There is a natural compact-open topology on $\widehat{T}$, generated by open subsets of the form $W(K,U) := \{f \in \widehat{T} : f(K) \subset U\}$, where $K \subset T$ are compact, $U \subset \mathbb{S}^1$ are open. We call $\widehat{T}$ the (topological) dual of T.

For inductive and projective topologies, we have the following general results concerning their duals.

Proposition 20. *Let $\{P_n\}_n$ be a projective system of Hausdorff topological groups with structural maps $\pi_{n,m} : P_n \to P_m$ and $\pi_n : \lim\limits_{\longleftarrow n} P_n \to P_n$. Assume that all π_n and $\pi_{n,m}$ are surjective and open, and that for any n, n', there exists an n'' such that $n'' \le n$ and $n'' \le n'$. Then, as topological groups,*

$$\widehat{\lim_{\longleftarrow n} P_n} \simeq \lim_{\longrightarrow n} \widehat{P_n}.$$

Proof. Denote by $\widehat{\pi}_{n,m} : \widehat{P_m} \to \widehat{P_n}$, $f_m \mapsto f_m \circ \pi_{n,m}$, the dual of $\pi_{n,m} : P_n \to P_m$. Then, for an element $\lim\limits_{\longrightarrow n} f_n \in \lim\limits_{\longrightarrow n} \widehat{P_n}$, we have $\widehat{\pi}_{n,m}(f_m) = f_n$, or equivalently, $f_m \circ \pi_{n,m} = f_n$. Hence, for an element $x = \lim\limits_{\longleftarrow n} x_n \in \lim\limits_{\longleftarrow n} P_n$, we have $f_m(x_m) = f_m(\pi_{n,m}(x_n)) = f_n(x_n)$ for sufficiently small m, n. Based on this, we define a natural map

$$\varphi : \varinjlim_n \widehat{P_n} \longrightarrow \widehat{\varprojlim_n P_n}, \qquad \varinjlim_n f_n \mapsto f$$

where $f : \varprojlim_n P_n \to \mathbb{S}^1$, $x = \varprojlim_n x_n \mapsto f_n(x_n)$. From the above discussion, f is well defined. Moreover, we have the following

Lemma 21. *(1) f is continuous. In particular, φ is well defined;*
(2) φ is a bijection;
(3) φ is continuous; and
(4) φ is open.

Proof. (1) Let U be an open subset of $\mathbb{S}^1$. If $x = \varprojlim_n x_n \in f^{-1}(U)$, $f_n(x_n) \in U$ for sufficiently small n. In particular, $x_n \in f_n^{-1}(U)$. On the other hand, since f_n is continuous, $f_n^{-1}(U)$ is open. So, $\varprojlim_n f_n^{-1}(U)$ is an open neighborhood of $x = \varprojlim_n x_n$. Note that $f(\varprojlim_n f_n^{-1}(U)) = f_n(f_n^{-1}(U)) = U$. Hence $\varprojlim_n f_n^{-1}(U) \subset f^{-1}(U)$. Consequently, f is continuous, and hence φ is well defined.

(2) To prove that φ is injective, we assume that $\varphi(\varinjlim_n g_n) =: g = f$. Thus $f_n(x_n) = g_n(x_n)$ for sufficiently small n and for all $x = \varprojlim_n x_n \in \varprojlim_n P_n$. Note that $\pi_{n,m}$ are surjective. So $f_n(x_n) = g_n(x_n)$ for all $x_n \in P_n$. This means that $f_n = g_n$ for sufficiently small n. Consequently, $\varinjlim_n f_n \equiv \varinjlim_n g_n$, and hence φ is injective.

To show that φ is surjective, let $f : \varprojlim_n P_n \to \mathbb{S}^1$ be a continuous map. Then, for any open subset $U \subset \mathbb{S}^1$ containing 1, $f^{-1}(U)$ is an open neighborhood of 0 in $\varprojlim_n P_n$. Hence, we may write $f^{-1}(U)$ as $f^{-1}(U) = \varprojlim_n P_n \cap \prod_\alpha K_n$ where $K_n \subset P_n$ are open subsets and $K_n = P_n$ for almost all n. By assumptions, for $n_1, \ldots, n_r$ such that $K_{n_i} = P_{n_i}$, there exists an N such that $N \leq n_i$. Then, $f(\operatorname{Ker} \pi_N) = 1$. So, $f(\operatorname{Ker} \pi_n) = 1$ for all $n \leq N$. Built on this, we define, for $n \geq N$, the maps $f_n : P_n \to \mathbb{S}^1, x_n \mapsto f(x)$ if $\pi_n(x) = x_n$. Note that $f(x)$ always make sense, since $\pi_n : \varprojlim_n P_n \to P_n$ is surjective. Moreover, f_n's are well defined. Indeed, if $y \in \varprojlim_n P_n$ such that $\pi_n(y) = x_n$, then $\pi_n(y) = x_n = \pi_n(x)$ for $n \leq N$. Hence $x - y \in \operatorname{Ker} \pi_n$. This implies that $f(y) = f(x)$. Clearly, by definition, $\varphi(\varinjlim_n f_n) = f$. So φ is surjective.

(3) and (4) are direct consequences of the bijectivity of φ. Indeed, to prove that φ is continuous, it suffices to show that for open subsets of $\widehat{\varprojlim_n P_n}$ in the form $\mathfrak{U} = W(K, V)$, $\varphi^{-1}(\mathfrak{U})$ is open in $\varinjlim_n \widehat{P_n}$, where K is a compact subset of $\varprojlim_n P_n$ and V is an open subset of $\mathbb{S}^1$. Since π_n are continuous, $K_n = \pi_n(K)$ are compact. In this way, we get a inductive system of open subsets $\{W(K_n, V)\}_n$. Set $U = \varinjlim_n W(K_n, V)$. Note that, by the bijectivity of φ, $f(U) = \mathfrak{U}$. This shows that f is continuous.

To prove φ is open, let U be an open subset of $\varinjlim_n \widehat{P_n}$ such that $\big(W(K_n, V)\big) = \iota_n^{-1}(U)$ for a compact subset K_n of P_n for any n. $K := \lim_{\longleftarrow_n} K_n$ is compact in $\varprojlim_n P_n$. Consequently, $W(K,0)$ is open in $\widehat{\varprojlim_n P_n}$. Note that, from the bijectivity of φ, we have $\varphi(\varinjlim_n W(K_n,V)) = W(K,V)$. So φ is open. This proves the lemma and hence also the proposition.

Next we treat inductive systems. By definition, an inductive system $\{D_n\}_n$ of Hausdorff topological groups is called *compact oriented*, if for any compact subset $K \subset \varinjlim_n D_n$, there exists an index n_0 such that $K \subset D_{n_0}$.

Proposition 22. *Let $\{D_n\}_n$ be a compact oriented inductive system of Hausdorff topological groups with structural maps $\iota_n : D_n \to \varinjlim_n D_n$ and $\iota_{n,n'} : D_n \to D_{n'}$. Assume that $\iota_{n,n'}$ are injective and closed, and that, for any n, n', there exists an n'' such that $n'' \geq n$ and $n'' \geq n'$. Then, as topological groups,*

$$\widehat{\varinjlim_n D_n} \simeq \varprojlim_n \widehat{D_n}\,.$$

Proof. To start with, we define a map

$$\psi : \varprojlim_n \widehat{D_n} \longrightarrow \widehat{\varinjlim_n D_n}, \qquad \varprojlim_n f_n \mapsto f$$

where $f : \varinjlim_n D_n \to \mathbb{S}^1$, $\varinjlim_n x_n \mapsto f_n(x_n)$.

Lemma 23. *(1) f is well defined and continuous. In particular, ψ is well defined.*
(2) ψ is a bijection;
(3) ψ is continuous; and
(4) ψ is open.

Proof. (1) Note that for $n < n'$, $f_n = f_{n'} \circ \iota_{n,n'}$ and $x_{n'} = \iota_{n,n'}(x_n)$. Consequently, $f_n(x_n) = f_{n'} \circ \iota_{n,n'}(x_n) = f_{n'}(\iota_{n,n'}(x_n)) = f_{n'}(x_{n'})$ for sufficiently large $n \leq n'$. So f is well defined.

To prove that f is continuous, let $U \subset \mathbb{S}^1$ be an open subset and take $\varinjlim_n x_n \in f^{-1}(U)$. By definition, $x_n \in f_n^{-1}(U) =: U_n$ and U_n are open. Hence $\mathfrak{U} := \varinjlim_n U_n$ is open and $x = \varinjlim_n x_n \in \mathfrak{U}$. Moreover, $f(\mathfrak{U}) \subset U$. So f s continuous.

(2) Assume that $\psi(\varprojlim_n f_n) = f \equiv 0$. Then, for any $x = \varinjlim_n x_n$, $f(x) = 0$. This means that for all n, and $x_n \in D_n$, $f_n(x_n) = f(x) = 0$ where x is determined by the condition that for all $n' \geq n$, $x_{n'} = \iota_{n,n'} x_n$. (Since ι_n are injective, this is possible.) Thus $f_n \equiv 0$ and hence $\varprojlim_n f_n = 0$.

For any $f \in \varprojlim_n \widehat{D_n}$, let $f_n = f \circ \iota_n : D_n \to \mathbb{S}^1$. Clearly, f_n is continuous. So $f_n \in \widehat{D_n}$. Moreover, for all $n' \geq n$, $f_n = f \circ \iota_n = f \circ \iota_{n'} \circ \iota_{n,n'} = f_{n'} \circ \iota_{n,n'}$. That is, $\{f_n\}_n$ forms a projective limit. Obviously, $\psi(\varprojlim_n f_n) = f$.

(3) This is rather involved: not only the just proved bijectivity of ψ, but all assumptions for our injective system are used here. Let $\mathfrak{U} = W(K,V)$ be an open subset of $\widehat{\varinjlim_n D_n}$, where $K \subset \varinjlim_n D_n$ is compact and V is an open subset of $\mathbb{S}^1$. By assumptions, for any n', $\iota_{n'}^{-1}(D_n)$ is closed. Hence $D_n \subset \varinjlim_n D_n$ are closed.

So $K_n := K \cap D_n$ are compact. If $\lim\limits_{\longleftarrow n} f_n$ is an element in $\psi^{-1}(\mathfrak{U}) \subset \widehat{\lim\limits_{\longrightarrow n} D_n}$, we have $f_n \in W(K_n, V)$, and $\lim\limits_{\longleftarrow n} f_n \in \lim\limits_{\longleftarrow n} W(K_n, V) = \psi^{-1}(W(K, V))$. So it suffices to show that $\lim\limits_{\longleftarrow n} W(K_n, V)$ is open. This is a direct consequence of our assumptions. Indeed, since our inductive system if compact oriented, there exists a certain n_0 such that $K = \lim\limits_{\longrightarrow n} K_n \subset D_{n_0}$. Hence, $K_n = K_{n_0} = K$ for all $n \geq n_0$. $\lim\limits_{\longleftarrow n} W(K_n, V) = \pi_{n_0}^{-1}(W(K_{n_0}, V))$. Hence, $\lim\limits_{\longleftarrow n} W(K_n, V)$ is open.

(4) This is a direct consequence of the bijectivity of ψ. Indeed, let $\mathfrak{U} \subset \lim\limits_{\longleftarrow n} \widehat{D_n}$ be an open subset. By definition, without loss of generality, we may assume that there exists an n and an open subset $W(K_n, V)$ of $\widehat{D_n}$ such that $\mathfrak{U} = \pi_n^{-1}(W(K_n, V))$. Since K_n is a compact subset of D_n and D_n is closed in $\lim\limits_{\longrightarrow n} D_n$, K_n is compact in $\lim\limits_{\longrightarrow n} D_n$. On the other hand, $\psi(\mathfrak{U}) = \psi(\pi_n^{-1}(W(K_n, V))) = W(K_n, V)$. So $\psi(\mathfrak{U})$ is open in $\widehat{\lim\limits_{\longrightarrow n} D_n}$. This proves the lemma and hence also the proposition.

3.1.2 Adelic spaces and their ind-pro topologies

Let X be an arithmetic surface. For a complete flag (X, C, x) on X (with C an irreducible curve on X and x a close point on C), let $k(X)_{C,x}$ its associated local ring. By Theorem 1, $k(X)_{C,x}$ is a direct sum of two dimensional local fields. Denote by $(\pi_C, t_{x,C})$ a local parameter defined by the flag C of X, and fix a Madunts-Zhukov lifting ([MZ])

$$h_C = (h_{\pi_C, t_{X,c}}) : \mathbb{A}_{C,01} \simeq \prod\nolimits'_{x:x\in C} \widehat{\mathcal{O}}_{C,x} / \pi_C \prod\nolimits'_{x:x\in C} \widehat{\mathcal{O}}_{C,x} \xrightarrow{\text{lifting}} \prod\nolimits'_{x:x\in C} \widehat{\mathcal{O}}_{C,x}$$

Then, following Parshin, see e.g., Example 2,

$$\begin{aligned}
\mathbb{A}_X^{\mathrm{fin}} = \mathbb{A}_{X,012} &= \prod_{x\in C}{}' k(X)_{C,x} := \prod_C{}' \Big(\prod_{x:x\in C} k(X)_{C,x} \Big) \\
&:= \left\{ \big(\sum_{i_C=-\infty}^{\infty} h_C(a_{i_C}) \pi_C^{i_C} \big)_C \in \prod_C \prod_{x:x\in C} k(X)_{C,x} \;\middle|\; \begin{array}{c} a_{i_C} \in \mathbb{A}_{C,01}, \\ a_{i_C} = 0 \ (i_C \ll 0); \\ D \min\{i_C : a_{i_C} \neq 0\} \geq 0 \ (\forall' C) \end{array} \right\}.
\end{aligned}$$

This gives the finite adelic space for X. Moreover, from Parshin-Osipov, see e.g., Definition 5, we have the infinite adelic space $\mathbb{A}_X^{\infty} := \mathbb{A}_{X_F} \widehat{\otimes}_{\mathbb{Q}} \mathbb{R}$, and hence the total arithmetic adelic space $\mathbb{A}_X^{\mathrm{ar}}$ for the arithmetic surface X:

$$\mathbb{A}_X^{\mathrm{ar}} := \mathbb{A}_X^{\mathrm{fin}} \oplus \mathbb{A}_X^{\infty}.$$

Moreover, there are natural ind-pro structures on $\mathbb{A}_X^{\mathrm{fin}}$, $\mathbb{A}_X^{\infty}$ and hence on $\mathbb{A}_X^{\mathrm{ar}}$, since

$$\begin{aligned}
\mathbb{A}_X^{\mathrm{fin}} &= \lim_{\longrightarrow D} \lim_{\longleftarrow E: E\leq D} \mathbb{A}_{X,12}(D) / \mathbb{A}_{X,12}(E), \\
\mathbb{A}_X^{\infty} &= \lim_{\longrightarrow D} \lim_{\longleftarrow E: E\leq D} \mathbb{A}_{X_F,1}(D) / \mathbb{A}_{X_F,1}(E) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R}.
\end{aligned}$$

Consequently, induced from the locally compact (Hausdorff) topologies on spaces $\mathbb{A}_{X,12}(E)/\mathbb{A}_{X,12}(E)$ and $\mathbb{A}_{X_F,1}(D)/\mathbb{A}_{X_F,1}(E)\widehat{\otimes}_{\mathbb{Q}}\mathbb{R}$, we get canonical ind-pro topologies on $\mathbb{A}_X^{\mathrm{fin}}$, $\mathbb{A}_X^{\infty}$, and hence on $\mathbb{A}_X^{\mathrm{ar}}$, which can be easily seen to be Hausdorff. For example, by [MZ], a fundamental system of open neighborhood of 0 in $\mathbb{A}_X^{\mathrm{fin}}$ is given by

$$\left\{ \Big(\sum_{i_C=-\infty}^{\infty} h_C(a_{i_C})\pi_C^{-i_C} \Big)_C \in \mathbb{A}_X^{\mathrm{fin}} : \begin{array}{r} a_{i_C} \in U_{i_C} \subset \mathbb{A}_{C,01} \text{ open subgroup} \\ U_{i_C} = \mathbb{A}_{C,01} \ \forall\ i_C \gg 0 \\ \min\{i_C : U_{i_C} = \mathbb{A}_{C,01}\} \leq 0\ (\forall' C) \end{array} \right\}. \quad (8)$$

We have a similar descriptions for $\mathbb{A}_X^{\infty}$. However, while, with respect to these canonical topologies, $\mathbb{A}_X^{\mathrm{fin}}$, $\mathbb{A}_X^{\infty}$, and $\mathbb{A}_X^{\mathrm{ar}}$ are additive topological groups, they are not topological rings. That is, the multiplication operations are not continuous for these spaces. Still, in §3.2.1, we will prove the following very useful

Proposition 24. *For a fixed $\mathbf{a} \in \mathbb{A}_C^{\mathrm{ar}}$, the scalar product by $\mathbf{a}$, namely, the map $\mathbb{A}_X^{\mathrm{ar}} \xrightarrow{\mathbf{a}\times} \mathbb{A}_X^{\mathrm{ar}},\ \mathbf{x} \mapsto \mathbf{ax}$, is continuous.*

Remark. This result can be used to establish a similar result for two dimensional local fields. Indeed, since $\prod'_{x\in C} k(X)_{C,x} = \lim\limits_{\longrightarrow n} \lim\limits_{\longleftarrow m:m\leq n} \mathbb{A}_{X,12}(nC)\big/\mathbb{A}_{X,12}(mC)$, as a subspace of $\mathbb{A}_X^{\mathrm{fin}} = \lim\limits_{\longrightarrow D} \lim\limits_{\longleftarrow E:E\leq D} \mathbb{A}_{X,12}(D)\big/\mathbb{A}_{X,12}(E)$, there is a natural ind-pro topology on $k(X)_{C,x}$, induced from the ind-pro topology on $\mathbb{A}_{X,012}$. Similarly, for a two dimensional local field F, we have $F = \lim\limits_{\longrightarrow n} \lim\limits_{\longleftarrow m:m\leq n} \mathfrak{m}_F^{-n}\big/\mathfrak{m}_F^{-m}$, where $\mathfrak{m}_F$ denotes the maximal ideal of F. So from the natural locally compact topologies on the quotient spaces $\mathfrak{m}_F^{-n}\big/\mathfrak{m}_F^{-m}$, we obtain yet another ind-pro topology on F, and hence on $k(X)_{C,x}$, since $k(X)_{C,x}$ is also a direct sum of two dimensional local fields. Induced from the same roots of locally compact topology on one-dimensional local fields, these two topologies on $k(X)_{C,x}$ are equivalent. This, with the above proposition, then proves the following

Corollary 25. *For a fixed $a_{C,x} \in k(X)_{C,x}$, the scalar product by $a_{C,x}$, namely, the map $k(X)_{C,x} \xrightarrow{a_{C,x}\cdot} k(X)_{C,x},\ \alpha \mapsto a_{C,x}\alpha$ is continuous. In particular, the scalar product of a fixed element on a two dimensional local field is continuous.*

We will use this result in an on-going work to prove that, with respect to the canonical ind-pro topology, two dimensional local fields are self-dual as topological groups.

3.1.3 Adelic spaces are complete

In this section, we show that adelic spaces $\mathbb{A}_X^{\mathrm{fin}}$ and $\mathbb{A}_X^{\infty}$ are complete. For basics of complete topological groups, please refer to [Bo] and [G]. We begin with

Proposition 26. *The subspaces $\mathbb{A}_{X,12}(D) \subset \mathbb{A}_X^{\mathrm{fin}}$ and $\mathbb{A}_{X_F,1}(D) \subset \mathbb{A}_X^{\infty}$, and the level two subspace $\mathbb{A}_{X,01}^{\mathrm{ar}}, \mathbb{A}_{X,02}^{\mathrm{ar}}$ and $\mathbb{A}_{X,12}^{\mathrm{ar}}(D)$ of $\mathbb{A}_X^{\mathrm{ar}}$ are complete and hence closed.*

Proof. As our proof below works for all other types as well, we only treat $\mathbb{A}_{X,12}(D) \subset \mathbb{A}_X^{\mathrm{fin}}$ to demonstrate. Since $\mathbb{A}_{X,12}(D)\big/\mathbb{A}_{X,12}(E)$'s are finite dimensional vector spaces over one dimensional local field, which is locally compact, so they are complete. Consequently, as a projective limit of complete spaces, $\mathbb{A}_{X,12}(D) = \lim\limits_{\longleftarrow E:E\leq D} \mathbb{A}_{X,12}(D)\big/\mathbb{A}_{X,12}(E)$ is complete. It is also closed since $\mathbb{A}_{X,012}$ is Hausdorff.

Proposition 27. *For an arithmetic surface X, its associated adelic spaces $\mathbb{A}_X^{\mathrm{fin}}$ and $\mathbb{A}_X^{\infty}$ are complete.*

Proof. We will give a uniform proof for both finite and infinite cases. For this reason, we use simply $\mathbb{A}$ to denote both $\mathbb{A}_X^{\mathrm{fin}}$ and $\mathbb{A}_X^{\infty}$, and $A(D)$ for both $\mathbb{A}_{X,12}(D)$ and $\mathbb{A}_{X_F}(D)\widehat{\otimes}_{\mathbb{Q}}\mathbb{R}$. Clearly, it suffices to prove the following

Lemma 28. *For a strictly increasing sequence $\{A(D_n)\}_n$ in $\mathbb{A}$, $\varinjlim_n A(D_n)$ is complete.*

Proof. Let $\{a_n\}_n$ be a Cauchy sequence of $\varinjlim_n A(D_n)$. We will show that these exists a divisor D such that $\{a_n\}_n \subset A(D)$. Assume that, on the contrary, for all divisors D, $\{a_n\}_n \not\subset A(D)$. We claim that then there exists a subsequence $\{a_{k_n}\}_n$ of $\{a_n\}$, a (strictly increasing) subsequence $\{D_{k_n}\}_n$ of $\{D_n\}_n$, and an open neighborhood U of 0 in $\varinjlim_n A(D_n)$ such that (i) $a_{k_n} \in A(D_{k_n})\backslash A(D_{k_{n-1}})$ for all $n \geq 2$; (ii) $a_{k_1}, \dots, a_{k_n}, \dots \notin U$ and (iii) $a_{k_{i+1}}, \dots, a_{k_n}, \dots \notin U + A(D_i)$, $i \geq 1$. If so, since $a_{k_m} \notin U + A(D_m)$ and $a_{k_n} \in A(D_m)$, for any $n > m$, $a_{k_n} - a_{k_m} \notin U$. So, $\{a_{k_n}\}_n$ is not a Cauchy sequence of $\varinjlim_n A(D_n)$. a contradiction. Therefore, there exists a divisor D such that $\{a_n\}_n \subset A(D)$. By Proposition 26, $A(D)$ is complete. So the Cauchy sequence $\{a_n\}_n$ is convergent in $A(D)$ and hence in $\varinjlim_n A(D_n)$ as well.

To prove the claim, we select $\{a_{k_n}\}_n$ and the corresponding D_{k_n}'s as follows. We begin with $a_{k_1} = a_1$. Being an element of $\mathbb{A}$, there always a divisor D_{k_1} such that $a_{k_1} \in A(D_{k_1})$. Since for all D, $\{a_n\}_n \not\subset A(D)$, there exists k_2 and a divisor D_{k_2} such that $D_{k_2} > D_{k_1}$ and $a_{k_2} \in A(D_{k_2}) - A(D_{k_1})$. By repeating this process, we obtain a subsequence $\{a_{k_n}\}_n$ of $\{a_n\}$, a (strictly increasing) subsequence $\{D_{k_n}\}_n$ of $\{D_n\}_n$ such that (i) above holds. Hence to verify the above claim, it suffices to find an open subset U satisfying the conditons (ii) and (iii) above. This is the contents of the following

Sublemma 29. *Let $\{A(D_n)\}_n$ be a strictly increasing sequence and $\{a_n\}_n$ be a sequence of elements of $\mathbb{A}$. Assume that $a_n \in A(D_n) - A(D_{n-1})$ for all $n \geq 1$. Then there exists an open subset U of $\varinjlim_n A(D_n)$ such that $a_1, \dots, a_n, \dots \notin U$ and $a_{m+1}, \dots, a_n, \dots \notin U + A(D_m)$ for all $m < n$.*

Proof. We separate the finite and infinite adeles.

Finite Adeles Since $A(D_1)/A(D_0)$ is Hausdorff, there exists an open, and hence closed, subgroup $U_1 \subset A(D_1)$ such that $a_1 \notin U_1$ and $U_1 \supset A(D_0)$. Since $A(D_1)$ is complete and U_1 is closed in $A(D_1)$, U_1 is complete as well. Now, viewing in $A(D_2)$, since $A(D_2)$ is Hausdorff, U_1 is a complete subspace, so U_1 is closed in $A(D_2)$. Hence $A(D_2)/U_1$ is Hausdorff too. Therefore, there exists an open and hence closed subgroup $V_{2,0}$ of $A(D_2)$ such that $a_1, a_2 \notin V_{2,0}$ and $V_{2,0} \supset U_1$. In addition, $A(D_2)/A(D_1)$ is Hausdorff, there exists an open subgroup $V_{2,1}$ such that $a_2 \notin V_{2,1}$ and $V_{2,1} \supset A(D_2)$. Consequently, if we set $U_2 = V_{2,0} \cap V_{2,1}$, U_2 is an open hence closed subgroup of $A(D_2)$ such that $a_1, a_2 \notin U_2$, $a_2 \notin U_2 + A(D_1)$ and $U_2 \supset U_1$. So, inductively, we may assume that there exists an increasing sequence of open subgroups $U_1, \dots, U_{n-1}$ satisfying the properties required. In particular, the following quotient spaces $A(D_n)/U_{n-1} + A(D_0)\big(= A(D_n)/U_{n-1}\big), \dots, A(D_n)/U_{n-1} +$

$A(D_m), \dots, A(D_n)/U_{n-1}+A(D_{n-1})\big(=A(D_n)/A(D_{n-1})\big)$ are Hausdorff. Hence, there are open subgroups $V_{n,m},\ 0 \leq m \leq n-1$ of $A(D_n)$ such that $a_{m+1}, \dots, a_n \notin V_{n,m}$ and $V_{n,m} \supset U_{n-1} + A(D_m)$. Define $U_n := \cap_{m=1}^{n-1} V_{n,m}$. Then U_n is an open subgroup of $A(D_n)$ satisfying $a_1, \dots, a_n \notin U_n$, $a_{m+1}, \dots, a_n \notin U_n + A(D_m),\ 1 \leq m \leq n-1$ and $U_n \supset U_{n-1}$. Accordingly, if we let $U = \varinjlim_n U_n$, by definition, U is an open subgroup of $\varinjlim_n A(D_n)$, and from our construction, $a_1, \dots, a_n, \dots \notin U$ and $a_{m+1}, \dots, a_n, \dots \notin U + A(D_m),\ m \geq 1$.

Infinite Adeles Since $A(D_1)$ is Hausdorff, there exists an open subset U_1 of $A(D_1)$ such that $a_1 \notin A(D_1)$. Moreover, since $A(D_2) \simeq A(D_2)/A(D_1) \oplus A(D_1)$ and $A(D_2)/A(D_1)$ is Hausdorff, there exists an open subset U_2 of $A(D_2)$ such that $a_1, a_2 \notin U_2$ and $U_2 \cap A(D_1) = U_1$. In particular, $a_2 \notin U_2 + A(D_1)$. Similarly, as above, with an inductive process, based on the fact that $A(D_n) \simeq A(D_n)/A(D_{n-1}) \oplus A(D_{n-1})$ and $A(D_n)/A(D_{n-1})$ is Hausdorff, there exists an open subset U_n of $A(D_n)$ such that $a_1, \dots, a_n \notin U_n$ and $U_n \cap A(D_n) = U_{n-1}$. Consequently, $a_{m+1}, \dots, a_n \notin U_n + A(D_m),\ 1 \leq m \leq n-1$. In this way, we obtain an infinite increasing sequence of open subsets U_n. Let $U = \varinjlim_n U_n$. Then by definition U is an open subset of $\varinjlim_n A(D_n)$ satisfying $a_1, \dots, a_n, \dots \notin U$ and $a_{m+1}, \dots, a_n, \dots \notin U + A(D_m),\ m \geq 1$. This then proves the sublemma, the lemma and hence also the proposition.

3.1.4 Adelic spaces are compact oriented

In this section, we show that adelic spaces $\mathbb{A}_X^{\mathrm{fin}}$ and $\mathbb{A}_X^{\infty}$ are compact oriented.

Proposition 30. *For an arithmetic surface X, its associated adelic spaces $\mathbb{A}_X^{\mathrm{fin}}$ and $\mathbb{A}_X^{\infty}$ are compact oriented. That is to say, for any compact subgroup, resp. a compact subset, K in $\mathbb{A}_X^{\mathrm{fin}}$, resp. $\mathbb{A}_X^{\infty}$, there exists a divisor D on X, resp., on X_F, such that $K \subset \mathbb{A}_{X,12}(D)$, resp., $K \subset \mathbb{A}_{X_F}(D)$.*

Proof. We treat both finite and infinite places simultaneously. Assume that for all divisors D, $K \not\subset A(D)$. Fix a suitable D_0. By our assumption, $K \not\subset A(D_0)$. Since $\mathbb{A}$ is Hausdorff and $A(D_0)$ is closed, there exists a certain divisor D_1 and an element $a_1 \in K$, such that $D_1 > D_0$, $a_1 \in A(D_1) \backslash A(D_0)$. Similarly, since $A(D_1)$ is closed, $\mathbb{A}/A(D_0)$ is Hausdorff, we can find an open subgroup $U_1' \subset \mathbb{A}/A(D_0)$ such that $a_1 + A(D_0) \not\subset \mathbb{A}/A(D_0)$. Consequently, there exists an open subgroup U_1 of $\mathbb{A}$ such that $a_1 \notin U_1$ and $U_1 \supset A(D_0)$. Now use $(D_1, \mathbb{A})$ instead of $(D_0, \mathbb{A})$, by repeating the above construction, we can find a divisor D_2, an open subgroup $U_2 \subset \mathbb{A}$ and an element $a_2 \in K \cap \big(A(D_2) \backslash A(D_1)\big)$ such that $D_2 > D_1$, $a_2 \notin U_2$, $U_2 \supset U_1$. In this way, we obtain a sequence of divisors D_n, a sequence of elements $a_n \in K \cap \big(A(D_n) \backslash A(D_{n-1})\big)$ and a sequence of open subgroups U_n such that $D_\alpha > D_{n-1}$, $a_n \notin U_n$, $U_n \supset U_{n-1}$. Let $U = \varinjlim_n U_n$. Then U is an open, and hence closed, subgroup of $\mathbb{A}$. Consequently, $K \cap U$ is compact. This is a contradiction. Indeed, since $a_1, \dots, a_n \notin U_n$ for all n, the open covering $\{U_n\}_n$ of $K \cap U$ admits no finite sub-covering. This completes the proof.

3.1.5 Double dual of adelic spaces

We here prove the following

Proposition 31. *As topological groups, we have*

$$\Big(\lim_{\longleftarrow E:E\leq D}\mathbb{A}_{X,12}(D)\big/\mathbb{A}_{X,12}(E)\Big)^\vee \simeq \lim_{\longrightarrow E:E\leq D}\Big(\mathbb{A}_{X,12}(D)\big/\mathbb{A}_{X,12}(E)\Big)^\vee,$$

$$\Big(\lim_{\longleftarrow E:E\leq D}\big(\mathbb{A}_{X_F}(D)\big/\mathbb{A}_{X_F}(E)\widehat{\otimes}_{\mathbb{Q}}\mathbb{R}\big)\Big)^\vee \simeq \lim_{\longrightarrow E:E\leq D}\Big(\mathbb{A}_{X_F}(D)\big/\mathbb{A}_{X_F}(E)\widehat{\otimes}_{\mathbb{Q}}\mathbb{R}\Big)^\vee;$$

and

$$\Big(\lim_{\longrightarrow D}\mathbb{A}_{X,12}(D)\Big)^\vee \simeq \lim_{\longleftarrow D}\Big(\mathbb{A}_{X,12}(D)\Big)^\vee,$$

$$\Big(\lim_{\longrightarrow D}\big(\mathbb{A}_{X_F}(D)\widehat{\otimes}_{\mathbb{Q}}\mathbb{R}\big)\Big)^\vee \simeq \lim_{\longleftarrow D}\Big(\mathbb{A}_{X_F}(D)\widehat{\otimes}_{\mathbb{Q}}\mathbb{R}\Big)^\vee.$$

where, $\mathbb{A}_{X_F}(D)\widehat{\otimes}_{\mathbb{Q}}\mathbb{R} := \lim_{\longleftarrow E:E\leq D}\Big(\mathbb{A}_{X_F}(D)\big/\mathbb{A}_{X_F}(E)\widehat{\otimes}_{\mathbb{Q}}\mathbb{R}\Big)$.

In particular,

$$\widehat{\mathbb{A}_X^{\mathrm{fin}}} \simeq \lim_{\longleftarrow D}\lim_{\substack{\longrightarrow E\\ E\leq D}}\Big(\mathbb{A}_{X,12}(D)\big/\mathbb{A}_{X,12}(E)\Big)^\vee,$$

$$\widehat{\mathbb{A}_X^{\infty}} \simeq \lim_{\longleftarrow D}\lim_{\substack{\longrightarrow E\\ E\leq D}}\Big(\mathbb{A}_{X_F}(D)\big/\mathbb{A}_{X_F}(E)\widehat{\otimes}_{\mathbb{Q}}\mathbb{R}\Big)^\vee.$$

Proof. We apply Proposition 20, resp. Proposition 22, to prove the first, resp., the second, pairs of homeomorphisms. We need to check the conditions there.

As above, we treat finite adeles and infinite adeles simultaneously. So we use $\mathbb{A}$ and $A(D)$ as in §3.1.3. Then $A(D)=\lim_{\longleftarrow E:E\leq D}A(D)\big/A(E)$. Now, for $E<E'$, $\pi_{D/E,D/E'}: A(D)\big/A(E)\to A(D)\big/A(E')$ is the natural quotient map. So $\pi_{D/E,D/E'}$ are surjective and open. Similarly, $\pi_{D,D/E}: A(D)\to A(D)\big/A(E)$ are surjective and open. So, by Proposition 20, we get the first group of two homeomorphisms for topological groups.

To treat the second group, recall that $A(D)\big/A(E)$ are complete. So, their projective limits $A(D)$'s are complete. This implies that $\mathbb{A}(D)$ are closed in $\mathbb{A}$, since $\mathbb{A}$ is Hausdorff. On the other hand, for $D<D'$, $\mathbb{A}_{X,12}(D)\subset\mathbb{A}_{X,12}(D')$. So the structural maps $\iota_{D,D'}:\mathbb{A}(D)\to\mathbb{A}(D')$ and $\iota: A(D)\to\mathbb{A}$ are injective and closed. Thus, by Proposition 22, it suffices to show that the inductive system $\{A(D)\}_D$ is compact oriented. This is simply the contents of §3.1.3-4. All this then completes our proof, since the last two homeomorphisms are direct consequences of previous four.

Corollary 32. *As topological groups,*

$$\widehat{\widehat{\mathbb{A}_X^{\mathrm{fin}}}} \simeq \mathbb{A}_X^{\mathrm{fin}}, \qquad \widehat{\widehat{\mathbb{A}_X^{\infty}}} \simeq \mathbb{A}_X^{\infty}, \qquad \text{and hence} \qquad \widehat{\widehat{\mathbb{A}_X^{\mathrm{ar}}}} \simeq \mathbb{A}_X^{\mathrm{ar}}.$$

Proof. Since $A(D)/A(E)$ are locally compact and hence they are self dual. Thus, to prove this double dual properties for our spaces, it suffices to check the conditions listed in Proposition 22 for inductive systems $\{\lim_{\longrightarrow E:E\leq D}\widehat{A(D)\big/A(E)}\}_E$ and in Proposition 20 for the projective system $\{\widehat{A(D)}\}_D$. With the above lengthy discussions, all this now becomes rather routine. For example, to verify that $\widehat{\mathbb{A}}$ is Hausdorff, we only need to recall that $\mathbb{S}^1$ is compact. Still, as careful examinations would help understand the essences of our proof above, we suggest ambitious readers to supply omitted details.

3.2 Adelic spaces and their duals

3.2.1 Continuity of scalar products

We here show that Proposition 24, namely, the scalar product maps on adelic spaces are continuous, even adelic spaces are not topological rings.

Proposition 24. *For a fixed element* $\mathbf{a}$ *of* $\mathbb{A}_X^{\mathrm{fin}}$, *resp. of* $\mathbb{A}_X^{\infty}$, *the induced scalar product map:* $\phi_{\mathbf{a}}^{\mathrm{fin}}:\mathbb{A}_X^{\mathrm{fin}}\xrightarrow{\mathbf{a}\times}\mathbb{A}_X^{\mathrm{fin}}$, *resp.,* $\phi_{\mathbf{a}}^{\infty}:\mathbb{A}_X^{\infty}\xrightarrow{\mathbf{a}\times}\mathbb{A}_X^{\infty}$, *is continuous.*

Proof. If $\mathbf{a}=0$, there is nothing to prove. Assume, from now on, that $\mathbf{a}=(a_C)_C=\big(\sum_{i_C=i_{C,0}}^{\infty}h_C(a_{i_C})\pi_C^{i_C}\big)_C\in\mathbb{A}_X^{\mathrm{fin}}\neq 0$. Here, for each C, we assume that $a_{i_{C,0}}\neq 0$. To prove that φ_a is continuous, by our description (8) of the ind-pro topology on $\mathbb{A}_X^{\mathrm{fin}}$, it suffices to show that for an open subgroup $U=(U_C)_C=\big(\sum_{j_C=-\infty}^{\infty}h_C(\mathbb{A}_{C,1}(D_{j_C}))\pi_C^{i_C}+\sum_{j_C=r_C}^{\infty}h_C(\mathbb{A}_{C,01})\pi_C^{i_C}\big)\cap\mathbb{A}_X^{\mathrm{fin}}$, as an open neighborhood of 0, its inverse image $\varphi_{\mathbf{a}}^{-1}(U)$ contains an open subgroup. For later use, set $I_C:=r_C-i_{C,0}$.

Let $\mathbf{b}=(b_C)_C=\big(\sum_{k_C=-\infty}^{\infty}h_C(b_{k_C})\pi_C^{k_C}\big)_C\in\varphi_{\mathbf{a}}^{-1}(U)\subset\mathbb{A}_X^{\mathrm{fin}}$. Then, for each fixed C, $a_Cb_C=\sum_{l_C=-\infty}^{\infty}\big(\sum_{i_C=i_{C,0}}^{\infty}h_C(a_{i_C})h_C(b_{l_C-i_C})\big)\pi_C^{l_C}$. Recall that h_C is the lifting map $h_C:\mathbb{A}_{C,01}\simeq\prod_{x:x\in C}{}'\widehat{\mathcal{O}}_{C,x}/\pi_C\prod_{x:x\in C}{}'\widehat{\mathcal{O}}_{C,x}\xrightarrow{\text{lifting}}\prod_{x:x\in C}{}'\widehat{\mathcal{O}}_{C,x}$. Thus if $b_{k_C}\in\mathbb{A}_{C,01}$, we always have $h_C(a_{i_C})h_C(b_{l_C-i_C})\in\sum_{m_C=0}^{\infty}h_C\big(\mathbb{A}_{C,01}\big)\pi_C^{m_C}$. Moreover, if we write, as we can, $a_{i_C}\in\mathbb{A}_{C,1}(F_{i_C})$, $b_{k_C}\in\mathbb{A}_{C,1}(E_{k_C})$ for some divisors F_{i_C} and E_{k_C}, we have $h_C(a_{i_C})h_C(b_{l_C-i_C})\in\sum_{m_C=0}^{\infty}h_C\big(\mathbb{A}_{C,1}(F_{i_C}+E_{l_C-i_C})\big)\pi_C^{m_C}$.

Now write $b_C=\big(\sum_{k_C=-\infty}^{I_C-1}+\sum_{k_C=I_C}^{\infty}\big)h_C(b_{k_C})\pi_C^{k_C}$. We will construct the required open subgroup according to the range of the degree index k_C.

(i) If $b_C\in\big(\sum_{k_C=I_C}^{\infty}h_C(\mathbb{A}_{C,01})\pi_C^{k_C}\big)\cap\big(\prod_{x:x\in C}{}'k(X)_{C,x}\big)$, we have $a_Cb_C\in U_C$;

(ii) To extend the range including also the degree I_C-1, choose a divisor E_{I_C-1} such that $h_C\big(\mathbb{A}_{C,1}(F_{i_{C,0}}+E_{I_C-1})\big)\subset h_C\big(\mathbb{A}_{C,1}(D_{r_C-1})\big)$. Then, if we choose $b_C\in\big(h_C\big(\mathbb{A}_{C,1}(E_{I_C-1})\pi_C^{I_C-1}+\sum_{k_C=I_C}^{\infty}h_C(\mathbb{A}_{C,01})\pi_C^{k_C}\big)\cap\big(\prod_{x:x\in C}{}'k(X)_{C,x}\big)$, we also have $a_Cb_C\in U_C$;

(iii) Similarly, to extend the range including the degree I_C-2, choose a divisor E_{I_C-2} such that $h_C\big(\mathbb{A}_{C,1}(F_{i_{C,0}}+E_{I_C-2})\big)\subset h_C\big(\mathbb{A}_{C,1}(D_{r_C-2})\big)\cap h_C\big(\mathbb{A}_{C,1}(D_{r_C-1})\big)$ and $h_C\big(\mathbb{A}_{C,1}(F_{i_{C,0}+1}+E_{I_C-2})\big)\subset h_C\big(\mathbb{A}_{C,1}(D_{r_C-1})\big)$. Then, if we choose $b_C\in\big(\sum_{k_C=I_C-2}^{I_C-1}h_C\big(\mathbb{A}_{C,1}(E_{k_C})\pi_C^{k_C}+\sum_{k_C=I_C}^{\infty}h_C(\mathbb{A}_{C,01})\pi_C^{k_C}\big)\cap\big(\prod_{x:x\in C}{}'k(X)_{C,x}\big)$, then we

have $a_C b_C \in U_C$.
Continuing this process repeatedly, we then obtain divisors E_{k_C}'s such that, for

$$b_C \in V_C := \Big(\sum_{k_C=-\infty}^{I_C-1} h_C\big(\mathbb{A}_{C,1}(E_{k_C})\pi_C^{k_C} + \sum_{k_C=I_C}^{\infty} h_C(\mathbb{A}_{C,01})\pi_C^{k_C}\Big) \cap \Big(\prod_{x:x\in C}{}' k(X)_{C,x}\Big),$$

we have $a_C b_C \in U_C$.

Since, for all but finitely many C, $r_C \leq 0$ and $i_{C,0} \geq 0$, or better, $I_C = r_C - i_{C,0} \leq 0$. Therefore, from above discussions, we conclude that $\prod_C V_C \cap \mathbb{A}_X^{\rm fin}$ is an open subgroup of $\mathbb{A}_X^{\rm fin}$ and $a\big(\prod_C V_C \cap \mathbb{A}_X^{\rm fin}\big) \subset U$. In particular, $\phi_{\mathbf{a}}$ is continuous.

A similar proof works for $\phi_{\mathbf{a}}^\infty$. We leave details to the reader.

3.2.2 Residue maps are continuous

Fix a non-zero rational differential ω on X. Then for an element $\mathbf{a}$ of $\mathbb{A}_X^{\rm fin}$, resp., $\mathbb{A}_X^\infty$, induced from the natural residue pairing $\langle\cdot,\cdot\rangle_\omega$, we get a natural map $\varphi_{\mathbf{a}}^{\rm fin} := \langle \mathbf{a}, \cdot\rangle_\omega : \mathbb{A}_X^{\rm fin} \longrightarrow \mathbb{R}/\mathbb{Z}$, resp., $\varphi_{\mathbf{a}}^\infty := \langle \mathbf{a}, \cdot\rangle_\omega : \mathbb{A}_X^\infty \longrightarrow \mathbb{R}/\mathbb{Z}$.

Lemma 33. *Let* $\mathbf{a}$ *be a fix element in* $\mathbb{A}_X^{\rm fin}$*, resp.,* $\mathbb{A}_X^\infty$*. Then the induced map* $\varphi_{\mathbf{a}}^{\rm fin} := \langle \mathbf{a}, \cdot\rangle_\omega : \mathbb{A}_X^{\rm fin} \longrightarrow \mathbb{R}/\mathbb{Z}$*, resp.,* $\varphi_{\mathbf{a}}^\infty := \langle \mathbf{a}, \cdot\rangle_\omega : \mathbb{A}_X^\infty \longrightarrow \mathbb{R}/\mathbb{Z}$*, is continuous. In particular, the residue map on arithmetic adeles* $\mathbb{A}_X^{\rm ar}$ *is continuous.*

Proof. We prove only for $\varphi_{\mathbf{a}}^{\rm fin}$, as a similar proof works $\varphi_{\mathbf{a}}^\infty$. Write $\mathbb{A}_X^{\rm fin} = \prod'_{\substack{F:\,2-\dim \\ \text{local field}}} F$. And, for each local field F, fix an element t_F of F such that for equal characteristic field F, t_F is a uniformizer of F, while for mixed characteristics field F, t_F is a lift of a uniformlzer of its residue field. Since, by Proposition 27, the scalar product is continuous, to prove the continuity of $\langle \mathbf{a}, \cdot\rangle_\omega$, it suffices to show that the residue map $\mathrm{Res} : \mathbb{A}_X^{\rm fin} \to \mathbb{R}/\mathbb{Z}$, $\mathbf{x} = (x_F) \mapsto \sum_F \mathrm{res}_F(x_F\, dt_F)$ is continuous. (Note that, by the definition of $\mathbb{A}_X^{\rm fin}$, see, e.g., §3.1.2, the above summation is a finite sum.) Since the open subgroup $\Big(\sum_{i_C=\infty}^{-1} h_C(\mathbb{A}_{C,1}(0))\pi_C^{i_C} + \sum_{i=0}^\infty h_C(\mathbb{A}_{C,01})\pi_C^{i_C}\Big) \cap \mathbb{A}_X^{\rm fin}$ is contained, the kernel of the residue map is an open subgroup. This proves the lemma.

3.2.3 Adelic spaces are self-dual

We will treat both $\mathbb{A}_X^{\rm fin}$ and $\mathbb{A}_X^\infty$ simultaneously. So as before, we use $\mathbb{A}$ to represent them.

Recall that, for a fixed $\mathbf{a} \in \mathbb{A}$, the map $\langle \mathbf{a}, \cdot\rangle_\omega : \mathbb{A} \to \mathbb{S}^1$ is continuous. Accordingly, we define a map $\varphi : \mathbb{A} \to \widehat{\mathbb{A}}$, $\mathbf{a} \mapsto \varphi_{\mathbf{a}} := \langle \mathbf{a}, \cdot\rangle_\omega$.

Proposition 34. *For the map* $\varphi : \mathbb{A} \to \widehat{\mathbb{A}}$, $\mathbf{a} \mapsto \varphi_{\mathbf{a}} := \langle \mathbf{a}, \cdot\rangle_\omega$*, we have*

(1) φ *is continuous;*

(2) φ *is injective;*

(3) The image of φ *is dense;*

(4) φ *is open.*

Proof. (1) For an open subset $W(K,0)$ of $\widehat{\mathbb{A}}$, where K is a compact subgroup, resp. a compact subset, of $\mathbb{A}$, let $U := \varphi^{-1}\big(W(K,0)\big)$. By Proposition 31, $\widehat{\mathbb{A}} = \varprojlim_D \varinjlim_{E:E\leq D} \widehat{A(D)/A(E)}$. So we may write $\chi_0 := \langle \mathbf{1}, \cdot\rangle_\omega$ as

$\lim_{\longleftarrow D} \lim_{\longrightarrow E: E \leq D} \chi_{D/E}$ with $\chi_{D/E} \in \widehat{A(D)/A(E)}$. Accordingly, write $A_{D/E} := A(D)/A(E)$, $K_{D/E} := K \cap A(D) \big/ K \cap A(E)$ and let $U_{D/E} := \big\{ a_{D/E} \in A_{D/E} : \chi_{D/E}\big(a_{D/E} K_{D/E}\big) = \{0\}$ resp. an open subset $V \big\}$. Since, for a fixed divisor D, $A(D)$ is closed in $\mathbb{A}$, $K \cap A(D)$ is a subgroup, resp. a subset, of $A(D)$. So, for $E \leq D$, $K_{D/E}$ is compact in $A_{D/E}$. Consequently, from the non-degeneracy of $\chi_{D/E}$ on locally compact spaces, $U_{D/E}$ is an open subgroup, resp., an open subset, of $\mathbb{A}$, and $U = \lim_{\longleftarrow D} \lim_{\longrightarrow E: E \leq D} U_{D/E}$. We claim that U is open. Indeed, by Proposition 30, $\mathbb{A}$ is compact oriented. So, for compact K, there exists a divisor D_1 such that $K \subset A(D_1)$. On the other hand, since χ_0 is continuous, there exists a divisor D_2 such that $A(D_1 + D_2) \subset \mathrm{Ker}(\chi_0)$. Hence $U \supset A(D_2)$. Thus, for a fixed D, with respect to sufficiently small $E \leq D$, we have $U_{D/E} = A_{D/E}$. This verifies that U is open, and hence proves (1), since the topology of $\widehat{\mathbb{A}}$ is generated by the open subsets of the form $W(K, 0)$.

(2) is a direct consequence of the non-degeneracy of the residue pairing. So we have (2).

To prove (3), we use the fact that $\mathbb{A} \overset{\psi}{\simeq} \widehat{\widehat{\mathbb{A}}}$, where, for $\mathbf{a} \in \mathbb{A}$, $\psi_{\mathbf{a}}$ is given by $\psi_{\mathbf{a}} : \widehat{\mathbb{A}} \to \mathbb{S}^1$, $\chi \mapsto \chi(\mathbf{a})$. Thus to show that the image of φ is dense, it suffices to show that the annihilator subgroup $\mathrm{Ann}\big(\mathrm{Im}(\varphi)\big)$ of $\mathrm{Im}(\varphi)$ is zero. Let then $\mathbf{x} \in \mathrm{Ann}\big(\mathrm{Im}(\varphi)\big)$ be an annihilator of $\mathrm{Im}(\varphi)$. Then, by definition, $\{0\} = \psi_{\mathbf{x}}\big(\{\varphi_{\mathbf{a}} : \mathbf{a} \in \mathbb{A}\}\big) = \{\varphi_{\mathbf{a}}(\mathbf{x}) : \mathbf{a} \in \mathbb{A}\}$. That is to say, $\langle \mathbf{a}, \mathbf{x} \rangle_\omega = 0$ for all $\mathbf{a} \in \mathbb{A}\}$. But the residue pairing is non-degenerate. So, $\mathbf{x} = 0$.

(4) This is the dual of (2). Indeed, let $U \subset \mathbb{A}$ be an open subgroup, resp. an open subset, of $\mathbb{A}$. Then $U \cap A(D)$ is open in $A(D)$. Since $A(D)$ is closed, $U_{D/E} := U \cap A(D) \big/ U \cap A(E)$ is open in $A_{D/E}$. This, together with the fact that $\chi_{D/E}$ is non-degenerate on its locally compact base space, implies that $K_{D/E} := \big\{ a_{D/E} \in A_{D/E} : \chi_{D/E}\big(a_{D/E} \cdot U_{D/E}\big) = \{0\}$ resp. an open subset $V \big\}$ is a compact subset, resp. a compact subset. Let $K := \lim_{\longrightarrow D} \lim_{\longleftarrow E: E \leq D} K_{D/E}$. Since U is open, there exists a divisor E such that $A(E) \subset U$. This implies that there exists a divisor D such that $K \subset A(D)$. Otherwise, assume that, for any D, $K \not\subset A(D)$. Then, there exists an element $k \in K$ such that $k \notin A(\omega) - E)$. Hence we have $\chi(k \cdot A(E)) \neq \{0\}$, a contradiction. This then completes the proof of (4), and hence the proposition.

We end this long discussions on ind-pro topology over adelic space $\mathbb{A}_X^{\mathrm{ar}}$ with the following main theorem.

Theorem 35. *Let X be an arithmetic surface. Then, as topological groups,*

$$\widehat{\mathbb{A}_X^{\mathrm{fin}}} \simeq \mathbb{A}_X^{\mathrm{fin}} \quad \text{and} \quad \widehat{\mathbb{A}_X^{\infty}} \simeq \mathbb{A}_X^{\infty}. \qquad \textit{In particular,} \quad \widehat{\mathbb{A}_X^{\mathrm{ar}}} \simeq \mathbb{A}_X^{\mathrm{ar}}.$$

Proof. With all the preparations above, this now becomes rather direct. Indeed, by Proposition 34, we have an injective continuous open morphism $\varphi : \mathbb{A} \to \widehat{\mathbb{A}}$. So it suffices to show that φ is surjective. But this is a direct consequence of the fact that φ is dense, since both $\mathbb{A}$ and $\widehat{\mathbb{A}}$ are complete and Hausdorff. This proves the theorem and hence also Theorem II.

3.2.4 Proof of cohomological duality

Now we are ready to prove Theorem 19, or the same Theorem III, for the duality of cohomology groups. Recall that for a non-zero rational differential ω, by Proposition 12, we have a non-degenerate residue pairing

$$\langle\cdot,\cdot\rangle_\omega : \mathbb{A}_X^{\mathrm{ar}} \times \mathbb{A}_X^{\mathrm{ar}} \longrightarrow \mathbb{S}^1.$$

Moreover, by Theorem II just proved, we obtain a natural homeomorphism of topological groups

$$\mathbb{A}_X^{\mathrm{ar}} \simeq \widehat{\mathbb{A}_X^{\mathrm{ar}}}, \quad \mathbf{a} \mapsto \langle \mathbf{a}, \cdot\rangle_\omega.$$

This, with a well-known argument which we omit, implies the following

Lemma 36. *With respect to the non-degenerate pairing $\langle\cdot,\cdot\rangle$ on $\mathbb{A}_X^{\mathrm{ar}}$, we have*
(i) If W_1 and W_2 are subgroups of $\mathbb{A}_X^{\mathrm{ar}}$,

$$(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp \qquad \text{and} \qquad (W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp;$$

(ii) If W is a closed subgroup of $\mathbb{A}_X^{\mathrm{ar}}$, then, algebraically and topologically,

$$(W^\perp)^\perp = W \qquad \text{and} \qquad W \simeq \widehat{\mathbb{A}_X^{\mathrm{ar}}/W^\perp}.$$

With this, we can complete our proof, using Proposition 15 for perpendicular subspaces of our level two subspaces $\mathbb{A}_{X,01}^{\mathrm{ar}}$, $\mathbb{A}_{X,02}^{\mathrm{ar}}$, $\mathbb{A}_{X,12}^{\mathrm{ar}}(D)$ of $\mathbb{A}_X^{\mathrm{ar}}$, as follows:
(1) **Topological duality between H^0_{ar} and H^2_{ar}**

$$\begin{aligned} H^2_{\mathrm{ar}}(\widehat{X,(\omega)} - D) &\simeq \Big(\mathbb{A}_{X,01}^{\mathrm{ar}} + \mathbb{A}_{X,02}^{\mathrm{ar}} + \mathbb{A}_{X,12}^{\mathrm{ar}}((\omega)-D)\Big)^\perp \\ &\simeq \Big(\mathbb{A}_{X,01}^{\mathrm{ar}}\Big)^\perp \cap \Big(\mathbb{A}_{X,02}^{\mathrm{ar}}\Big)^\perp \cap \Big(\mathbb{A}_{X,12}^{\mathrm{ar}}((\omega)-D)\Big)^\perp \\ &= \mathbb{A}_{X,01}^{\mathrm{ar}} \cap \mathbb{A}_{X,02}^{\mathrm{ar}} \cap \mathbb{A}_{X,12}^{\mathrm{ar}}(D) \simeq H^0_{\mathrm{ar}}(D); \end{aligned}$$

(2) **Topological duality among H^1_{ar}**

$$\begin{aligned} H^1_{\mathrm{ar}}(\widehat{X,(\omega)} - D) &= \Big(\frac{\mathbb{A}_{X,02}^{\mathrm{ar}} \cap \big(\mathbb{A}_{X,01}^{\mathrm{ar}} + \mathbb{A}_{X,12}^{\mathrm{ar}}((\omega)-D)\big)}{\mathbb{A}_{X,01}^{\mathrm{ar}} \cap \mathbb{A}_{X,02}^{\mathrm{ar}} + \mathbb{A}_{X,02}^{\mathrm{ar}} \cap \mathbb{A}_{X,12}^{\mathrm{ar}}((\omega)-D)}\Big)^{\widehat{}} \\ &\simeq \frac{\big(\mathbb{A}_{X,01}^{\mathrm{ar}} \cap \mathbb{A}_{X,02}^{\mathrm{ar}}\big)^\perp \cap \big(\mathbb{A}_{X,02}^{\mathrm{ar}} \cap \mathbb{A}_{X,12}^{\mathrm{ar}}((\omega)-D)\big)^\perp}{\big(\mathbb{A}_{X,02}^{\mathrm{ar}}\big)^\perp + \big(\mathbb{A}_{X,01}^{\mathrm{ar}} + \mathbb{A}_{X,12}^{\mathrm{ar}}((\omega)-D)\big)^\perp} \\ &= \frac{\big(\mathbb{A}_{X,01}^{\mathrm{ar}} + \mathbb{A}_{X,02}^{\mathrm{ar}}\big) \cap \big(\mathbb{A}_{X,02}^{\mathrm{ar}} + \mathbb{A}_{X,12}^{\mathrm{ar}}(D)\big)}{\mathbb{A}_{X,02}^{\mathrm{ar}} + \mathbb{A}_{X,01}^{\mathrm{ar}} \cap \mathbb{A}_{X,12}^{\mathrm{ar}}(D)} \\ &\simeq \frac{\big(\mathbb{A}_{X,01}^{\mathrm{ar}} + \mathbb{A}_{X,02}^{\mathrm{ar}}\big) \cap \mathbb{A}_{X,12}^{\mathrm{ar}}(D)}{\mathbb{A}_{X,01}^{\mathrm{ar}} \cap \mathbb{A}_{X,12}^{\mathrm{ar}}(D) + \mathbb{A}_{X,02}^{\mathrm{ar}} \cap \mathbb{A}_{X,12}^{\mathrm{ar}}(D)} \simeq H^1_{\mathrm{ar}}(X,D). \end{aligned}$$

This then completes the proof of Theorem III.

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REFERENCES

[B] A.A. Beilinson, Residues and adeles, Funct. Anal. Pril., 14 (1980), no. 1, 44-45; English transl. in Func. Anal. Appl., 14, no. 1 (1980), 34-35.

[Bo] N. Bourbaki, *Elements of Mathematics: General Topology*, Springer, 1987

[G] S.A. Gaal, *Point set topology*, Pure and Applied Mathematics, Vol. XVI, Academic Press,1964

[H] A. Huber, On the Parshin-Beilinson adeles for schemes, Abh. Math. Sem. Univ. Hamburg, 61 (1991), 249-273.

[Iw] K. Iwasawa, *Algebraic Theory of Algebraic Functions,* Lecture notes at Princeton university, 1975

[L] S. Lang, *Introduction to Arakelov theory*, Springer Verlag, 1988

[MZ] A. I. Madunts and I. B. Zhukov, Multidimensional complete fields: topology and other basic constructions, Trudy S.-Peterb. Mat. Obshch. (1995); English translation in Amer. Math. Soc. Transl. (Ser. 2) 165 (1995), 1-34.

[M1] M. Morrow, An explicit approach to residues on and dualizing sheaves of arithmetic surfaces. New York J. Math. 16 (2010), 575-627.

[M2] M. Morrow, Grothendieck's trace map for arithmetic surfaces via residues and higher adeles, Journal of Algebra and Number Theory, 6 (2012) 1503-1536

[OP] D. V. Osipov, A. N. Parshin, Harmonic analysis on local fields and adelic spaces. II. Izv. Ross. Akad. Nauk Ser. Mat. 75 (2011), no. 4, 91–164

[P1] A.N. Parshin, On the arithmetic of two-dimensional schemes. I. Distributions and residues. Izv. Akad. Nauk SSSR Ser. Mat. 40 (1976), no. 4, 736-773

[P2] A.N. Parshin, Chern classes, adeles and L-functions. J. Reine Angew. Math. 341 (1983), 174-192.

[S] J.P. Serre, Algebraic groups and class fields, GTM 117, 1988, Springer-Verlag

[SW] K. Sugahara and L. Weng, Topological duality of two dimensional local fields, in preparations

[T] J. Tate, Fourier analysis in number fields, and Hecke's zeta-functions. 1967 Algebraic Number Theory pp. 305-347 Thompson, Washington, D.C.

[W] L. Weng, Geometry of numbers, arXiv:1102.1302

[Y] A. Yekutieli, *An Explicit construction of the Grothendieck residue complex*, with an appendix by P. Sastry, Asterisque No. 208 (1992)

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