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NON-ABELIAN ZETA FUNCTIONS FOR FUNCTION FIELDS

By Lin Weng

Abstract. In this paper we initiate a geometrically oriented construction of non-abelian zeta functions for curves defined over finite fields. More precisely, we first introduce new yet genuine non-abelian zeta functions for curves defined over finite fields, by a "weighted count" on rational points over the corresponding moduli spaces of semi-stable vector bundles using moduli interpretation of these points. Then we define non-abelian L-functions for curves over finite fields using integrations of Eisenstein series associated to L^2-automorphic forms over certain generalized moduli spaces.

Introduction. In this paper we initiate a geometrically oriented construction of non-abelian zeta functions for curves defined over finite fields. It consists of two chapters.

More precisely, in Chapter I, we first introduce new yet genuine non-abelian zeta functions for curves defined over finite fields. This is achieved by a "weighted count" on rational points over the corresponding moduli spaces of semi-stable vector bundles using moduli interpretation of these points. We justify our construction by establishing basic properties for these new zetas such as functional equation and rationality, and show that if only line bundles are involved, our newly defined zetas coincide with Artin's Zeta. All this, in particular, the rationality, then leads naturally to our definition of (global) non-abelian zeta functions (for curves defined over number fields), which themselves are justified by a convergence result. We end this chapter with a detailed study on rank two non-abelian zeta functions for genus two curves, based on what we call infinitesimal structures of Brill-Noether loci (and Weierstrass points).

In Chapter II, we begin with a similar construction for the field of rationals to motivate what follows. In particular, we show that there is an intrinsic relation between our non-abelian zeta functions and Eisenstein series. Due to this, instead of introducing general non-abelian L-functions for curves defined over finite fields with more general test functions (as what Tate did in his Thesis for abelian L-functions), we then define non-abelian L-functions for curves over finite fields as integrations of Eisenstein series associated to L^2-automorphic forms over certain generalized moduli spaces. Here geometric truncations play a key role. Basic properties for these non-abelian L-functions, such as meromorphic continuation,

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functional equations and singularities, are established as well, based on the theory of Eisenstain series of Langlands and Morris. We end this chapter by establishing a closed formula for what we call the abelian parts of non-abelian L-functions associated with Eisenstein series for cusp forms, via the Rankin-Selberg method, motivated by a formula of Arthur and Langlands.

This work is an integrated part of our vast yet still developing Program for Geometric Arithmetic [We1], and is motivated by our new non-abelian L-functions for number fields [We2] in connection with non-abelian arithmetic aspects of global fields.

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Chapter I. Non-abelian zeta functions. This consists of two aspects: construction and justification. For the construction, we first introduce a new type of zeta functions for curves defined over finite fields using the corresponding moduli spaces of semi-stable vector bundles. We show that these new zeta functions are indeed rational and satisfy certain functional equations, based on the vanishing theorem (duality, Riemann-Roch theorem), for cohomologies of semi-stable vector bundles. Based on this, in particular, the rationality, we then introduce global non-abelian zeta functions for curves defined over number fields, via the Euler product formalism. Moreover, we establish a convergence result for our Euler products using the Clifford Lemma, an ugly yet quite explicit formula for local non-abelian zeta functions, a result of (Harder-Narasimhan) Siegel about quadratic forms, and Weil's theorem on the Riemann Hypothesis for Artin zeta functions.

As for the justification, we check that when only line bundles are involved (so moduli spaces of semi-stable bundles are nothing but the standard Picard groups), our (new) zeta functions, global and local, coincide with the classical Artin zeta functions for curves defined over finite fields and Hasse-Weil zeta functions for curves defined over number fields respectively. Moreover, as concrete examples, we compute rank two zeta functions for genus two curves by studying Weierstrass points and non-abelian Brill-Noether loci in terms of what we call their infinitesimal structures.

I.1 Local non-abelian zeta functions for curves. In this section, we introduce our non-abelian zeta functions for curves defined over finite fields. Basic properties for these non-abelian zeta functions, such as meromorphic extensions, rationality and functional equations, are established.

1.1. Moduli spaces of semi-stable bundles.

1.1.1. Semi-stable bundles. Let C be a regular, reduced and irreducible projective curve defined over an algebraically closed field $\bar{k}$. Then according to

Mumford [Mu], a vector bundle V on C is called semi-stable (resp. stable) if for any proper subbundle V' of V,

$$\mu(V') := \frac{d(V')}{r(V')} \leq (\text{resp.} <) \frac{d(V)}{r(V)} =: \mu(V).$$

Here d denotes the degree and r denotes the rank.

PROPOSITION. *Let V be a vector bundle over C. Then:* (a) *([HN]) there exists a unique filtration of subbundles of V, the Harder-Narasimhan filtration of V,*

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{s-1} \subset V_s = V$$

such that all V_i/V_{i-1} are semi-stable and for $1 \leq i \leq s-1$, $\mu(V_i/V_{i-1}) > \mu(V_{i+1}/V_i)$;

(b) *(see e.g.* [Se]*) if moreover V is semi-stable, there exists a filtration of subbundles of V, a Jordan-Hölder filtration of V,*

$$\{0\} = V^{t+1} \subset V^t \subset \cdots \subset V^1 \subset V^0 = V$$

such that for all $0 \leq i \leq t$, V^i/V^{i+1} is stable and $\mu(V^i/V^{i+1}) = \mu(V)$. Moreover, the associated graded bundle $Gr(V) := \oplus_{i=0}^t V^i/V^{i+1}$, the (Jordan-Hölder) graded bundle of V, is determined uniquely by V.

1.1.2. Moduli space of stable bundles. Following Seshadri, two semi-stable vector bundles V and W are called S-equivalent, if their associated Jordan-Hölder graded bundles are isomorphic, i.e., $\mathrm{Gr}(V) \simeq \mathrm{Gr}(W)$. Applying Mumford's general result on geometric invariant theory, Narasimhan and Seshadri proved the following:

THEOREM. (See e.g. [NS] and [Se].) *Let C be a regular, reduced, irreducible projective curve of genus $g \geq 2$ defined over an algebraically closed field. Then over the set $\mathcal{M}_{C,r}(d)$ (resp. $\mathcal{M}_{C,r}(L)$) of S-equivalence classes of rank r and degree d (resp. rank r and determinant L) semi-stable vector bundles over C, there is a natural normal, projective $(r^2(g-1)+1)$-dimensional (resp. $(r^2-1)(g-1)$-dimensional) algebraic variety structure.*

Remark. In this paper, we always assume that the genus of g is at least 2. For elliptic curves, whose associated moduli spaces are very special, please see [We3].

1.1.3. Rational points. Now assume that C is defined over a finite field k. It makes sense to talk about k-rational bundles over C, i.e., bundles which are defined over k. Moreover, from geometric invariant theory, projective varieties $\mathcal{M}_{C,r}(d)$ are defined over a certain finite extension of k; and if L itself is defined over k, the same holds for $\mathcal{M}_{C,r}(L)$. Thus it makes sense to talk about k-rational points of these moduli spaces too. The relation between these two types of rationality is given by Harder-Narasimhan based on a discussion about Brauer groups:

PROPOSITION. [HN] *Let C be a regular, reduced, irreducible projective curve of genus $g \geq 2$ defined over a finite field k. Then there exists a finite field $\mathbb{F}_q$ such that for all d (resp. all k-rational line bundles L), the subset of $\mathbb{F}_q$-rational points of $\mathcal{M}_{C,r}(d)$ (resp. $\mathcal{M}_{C,r}(L)$) consists exactly of all S-equivalence classes of $\mathbb{F}_q$-rational bundles in $\mathcal{M}_{C,r}(d)$ (resp. $\mathcal{M}_{C,r}(L)$).*

From now on, without loss of generality, we always assume that the finite fields $\mathbb{F}_q$ (with q elements) satisfy the property stated in the Proposition. Also for simplicity, we write $\mathcal{M}_{C,r}(d)$ (resp. $\mathcal{M}_{C,r}(L)$) for $\mathcal{M}_{C,r}(d)(\mathbb{F}_q)$ (resp. $\mathcal{M}_{C,r}(L)(\mathbb{F}_q)$), the subset of $\mathbb{F}_q$-rational points, and call them moduli spaces by an abuse of notations. Clearly these sets are all finite.

1.2. Local non-abelian zeta functions.

1.2.1. Definition. Let C be a regular, reduced, irreducible projective curve of genus $g \geq 2$ defined over the finite field $\mathbb{F}_q$ with q elements. Define the *rank r non-abelian zeta function $\zeta_{C,r,\mathbb{F}_q}(s)$ of C* by setting

$$\zeta_{C,r,\mathbb{F}_q}(s) := \sum_{V\in[V]\in\mathcal{M}_{C,r}(d),d\geq 0} \frac{q^{h^0(C,V)}-1}{\#\mathrm{Aut}(V)} \cdot (q^{-s})^{d(V)}, \qquad \mathrm{Re}(s) > 1.$$

PROPOSITION. *With the same notation as above, $\zeta_{C,1,\mathbb{F}_q}(s)$ is nothing but the classical Artin zeta function $\zeta_C(s)$ for curve C. That is to say,*

$$\zeta_{C,1,\mathbb{F}_q}(s) = \sum_{D\geq 0} \frac{1}{N(D)^s} =: \zeta_C(s) \qquad Re(s) > 1.$$

Here D runs over all effective divisors of C, and $N(D) := q^{d(D)}$ with $d(\Sigma_P n_P P) := \Sigma_P n_P d(P)$.

Proof. By definition, the classical Artin zeta function ([A], [Mo]) for C is given by

$$\zeta_C(s) := \sum_{D\geq 0} \frac{1}{N(D)^s}.$$

Thus by first grouping effective divisors according to their rational equivalence classes $\mathcal{D}$, then taking the sum on effective divisors in the same class, we obtain

$$\zeta_C(s) = \sum_{\mathcal{D}} \sum_{D\in\mathcal{D},D\geq 0} \frac{1}{N(D)^s}.$$

Clearly,

$$\sum_{D\in\mathcal{D},D\geq 0} \frac{1}{N(D)^s} = \frac{q^{h^0(C,\mathcal{D})}-1}{q-1} \cdot (q^{-s})^{d(\mathcal{D})}.$$

Therefore,

$$\zeta_C(s) = \sum_{L\in \operatorname{Pic}^d(C),d\geq 0} \frac{q^{h^0(C,L)}-1}{\#\mathrm{Aut}(L)} \cdot (q^{-s})^{d(L)}$$

due to the fact that $\mathrm{Aut}(L) \simeq \mathbb{F}_q^*$.

Remark. Before going further, let us explain the notation $V \in [V]$ appeared in the summation in detail. By $\sum_{V\in[V]}$, we mean that the sum is taken over all (isomorphism classes of) rational vector bundles V in $[V]$. From Prop. (b) in 1.1.1, for each fixed $[V]$, there are only finitely many terms involved. On the other hand, we may instead use only a single element V for each class $[V]$, say, one with maximal automorphism group (as used in the proof of the projectivity of moduli spaces). However, while interesting, such a change yields quite different functions. (See e.g. [We1].) Our decision to use all rational elements in $[V]$ is motivated by an adelic consideration, in particular, by Harder-Narasimhan's understanding of Siegel's formula.

1.2.2. Convergence and rationality. At this point, we must show that for general r, the infinite summation in the definition of our non-abelian zeta function $\zeta_{C,r,\mathbb{F}_q}(s)$ converges when $\mathrm{Re}(s) > 1$. For this, let us start with the following simple vanishing result for semi-stable vector bundles.

LEMMA 1. *Let V be a rank r semi-stable vector bundle of degree d on C. Then:*
(a) *if $d \geq r(2g-2)+1$, $h^1(C,V) = 0$;*
(b) *if $d < 0$, $h^0(C,V) = 0$.*

Proof. This is a direct consequence of the fact that if V and W are semi-stable vector bundles with $\mu(V) > \mu(W)$, then $H^0(C,\mathrm{Hom}(V,W)) = \{0\}$.

Thus, by definition,

$$\begin{aligned}\zeta_{C,r,\mathbb{F}_q}(s) \;=\; &\sum_{V\in[V]\in\mathcal{M}_{C,r}(d),0\leq d\leq r(2g-2)} \frac{q^{h^0(C,V)}-1}{\#\mathrm{Aut}(V)} \cdot (q^{-s})^{d(V)}\\ &+ \sum_{V\in[V]\in\mathcal{M}_{C,r}(d),d\geq r(2g-2)+1} \frac{q^{d(V)-r(g-1)}-1}{\#\mathrm{Aut}(V)} \cdot (q^{-s})^{d(V)}.\end{aligned}$$

Clearly only finitely many terms appear in the first summation, so it suffices to show that when $\mathrm{Re}(s) > 1$, the second term converges. For this purpose, we introduce what we call the Harder-Narasimhan numbers

$$\beta_{C,r,\mathbb{F}_q}(d) := \sum_{V\in[V]\in\mathcal{M}_{C,r}(d)} \frac{1}{\#\mathrm{Aut}(V)}.$$

LEMMA 2. *With the same notation as above, for all $n \in \mathbb{Z}$,*

$$\beta_{C,r,\mathbb{F}_q}(d+rn) = \beta_{C,r,\mathbb{F}_q}(d).$$

Proof. This comes from the following two facts: (1) there is a degree one $\mathbb{F}_q$-rational line bundle A on C; and (2) $\mathrm{Aut}(V) \simeq \mathrm{Aut}(V\otimes A^{\otimes n})$ and $d(V\otimes A^{\otimes n}) = d(V) + rn$.

Therefore, the second summation becomes

$$\begin{aligned}\sum_{i=1}^{r} \beta_{C,r,\mathbb{F}_q}(i) \sum_{n=2g-2}^{\infty} &\left(q^{nr+i-r(g-1)} - 1\right)\cdot (q^{-s})^{nr+i} \\ &= \sum_{i=1}^{r} \beta_{C,r,\mathbb{F}_q}(i)\cdot (q^{-s})^{i}\cdot \left(q^{i-r(g-1)}\cdot \frac{q^{(1-s)\cdot r(2g-2)}}{1-q^{(1-s)\cdot r}} - \frac{q^{(-s)\cdot r(2g-2)}}{1-q^{(-s)\cdot r}}\right),\end{aligned}$$

provided that $|q^{-s}| < 1$. Thus we have proved the following:

PROPOSITION. *The non-abelian zeta function $\zeta_{C,r,\mathbb{F}_q}(s)$ is well-defined for $Re(s) > 1$, and admits a meromorphic extension to the whole complex s-plane.*

Moreover, if we set $t := q^{-s}$ and introduce the non-abelian Z-function of C by

$$Z_{C,r,\mathbb{F}_q}(t) := \sum_{V\in[V]\in\mathcal{M}_{C,r}(d),d\geq 0} \frac{q^{h^0(C,V)}-1}{\#\mathrm{Aut}(V)}\cdot t^{d(V)}, \qquad |t| < 1.$$

Then the above calculation implies that

$$\begin{aligned}Z_{C,r,\mathbb{F}_q}(t) \;=\; &\sum_{d=0}^{r(2g-2)} \left(\sum_{V\in[V]\in\mathcal{M}_{C,r}(d)} \frac{q^{h^0(C,V)}-1}{\#\mathrm{Aut}(V)}\right)\cdot t^{d} \\ &+\sum_{i=1}^{r} \beta_{C,r,\mathbb{F}_q}(i)\cdot \left(\frac{q^{r(g-1)+i}}{1-q^r t^r} - \frac{1}{1-t^r}\right)\cdot t^{r(2g-2)+i}.\end{aligned}$$

Therefore, there exists a polynomial $P_{C,r,\mathbb{F}_q}(s) \in \mathbb{Q}[t]$ such that

$$Z_{C,r,\mathbb{F}_q}(t) = \frac{P_{C,r,\mathbb{F}_q}(t)}{(1-t^r)(1-q^r t^r)}.$$

In this way, we have established the following:

RATIONALITY. *Let C be a regular, reduced irreducible projective curve defined over $\mathbb{F}_q$ with $Z_{C,r,\mathbb{F}_q}(t)$ the rank r non-abelian Z-function. Then, there exists a polynomial $P_{C,r,\mathbb{F}_q}(s) \in \mathbb{Q}[t]$ such that*

$$Z_{C,r,\mathbb{F}_q}(t) = \frac{P_{C,r,\mathbb{F}_q}(t)}{(1-t^r)(1-q^r t^r)}.$$

1.2.3. Functional equation. To understand $P_{C,r,\mathbb{F}_q}(s)$ better, as well as for theoretical purpose, we next study functional equation for rank r zeta functions. Let us introduce the rank r non-abelian ξ-function $\xi_{C,r,\mathbb{F}_q}(s)$ by setting

$$\xi_{C,r,\mathbb{F}_q}(s) := \zeta_{C,r,\mathbb{F}_q}(s) \cdot (q^s)^{r(g-1)}.$$

That is to say,

$$\xi_{C,r,\mathbb{F}_q}(s) = \sum_{V \in [V] \in \mathcal{M}_{C,r}(d), d \geq 0} \frac{q^{h^0(C,V)} - 1}{\#\mathrm{Aut}(V)} \cdot (q^{-s})^{\chi(C,V)}, \qquad \mathrm{Re}(s) > 1,$$

where $\chi(C, V)$ denotes the Euler-Poincaré characteristic of V.

FUNCTIONAL EQUATION. *Let C be a regular, reduced irreducible projective curve defined over $\mathbb{F}_q$ with $\xi_{C,r,\mathbb{F}_q}(s)$ its associated rank r non-abelian ξ-function. Then,*

$$\xi_{C,r,\mathbb{F}_q}(s) = \xi_{C,r,\mathbb{F}_q}(1-s).$$

Before proving the functional equation, we give the following:

COROLLARY *With the same notation as above:*

(a) $P_{C,r,\mathbb{F}_q}(t) \in \mathbb{Q}[t]$ *is a degree $2rg$ polynomial;*

(b) *Denote all reciprocal roots of $P_{C,r,\mathbb{F}_q}(t)$ by $\omega_{C,r,\mathbb{F}_q}(i), i = 1, \ldots, 2rg$. Then after a suitable rearrangement,*

$$\omega_{C,r,\mathbb{F}_q}(i) \cdot \omega_{C,r,\mathbb{F}_q}(2rg - i) = q, \qquad i = 1, \ldots, rg;$$

(c) *For each $m \in \mathbb{Z}_{\geq 1}$, there exists a rational number $N_{C,r,\mathbb{F}_q}(m)$ such that*

$$Z_{r,C,\mathbb{F}_q}(t) = P_{C,r,\mathbb{F}_q}(0) \cdot \exp\left(\sum_{m=1}^{\infty} N_{C,r,\mathbb{F}_q}(m)\frac{t^m}{m}\right).$$

Moreover,

$$N_{C,r,\mathbb{F}_q}(m) = \begin{cases} r(1+q^m) - \sum_{i=1}^{2rg} \omega_{C,r,\mathbb{F}_q}(i)^m, & r\mid m; \\ -\sum_{i=1}^{2rg} \omega_{C,r,\mathbb{F}_q}(i)^m, & r \nmid m; \end{cases}$$

(d) *For any $a \in \mathbb{Z}_{>0}$, denote by ζ_a a primitive ath root of unity and set $T = t^a$. Then*

$$\prod_{i=1}^{a} Z_{C,r}(\zeta_a^i t) = (P_{C,r,\mathbb{F}_q}(0))^a \cdot \exp\left(\sum_{m=1}^{\infty} N_{r,C,\mathbb{F}_q}(ma)\frac{T^m}{m}\right).$$

Proof. (a) and (b) are direct consequences of the functional equation, while (c) and (d) are direct consequences of (a), (b) and the following well-known relations

$$\sum_{i=1}^{a} (\zeta_a^i)^m = \begin{cases} a, & a\mid m, \\ 0, & a \nmid m. \end{cases}$$

1.2.4. Proof of the functional equation. To understand the structure of the functional equation explicitly, we decompose the non-abelian ξ-function for curves. For this purpose, first recall that the canonical line bundle K_C of C is defined over $\mathbb{F}_q$. Thus, for all $n \in \mathbb{Z}$, we obtain the following natural $\mathbb{F}_q$-rational isomorphisms:

$$\begin{array}{rcl} \mathcal{M}_r(L) & \to & \mathcal{M}_r(L \otimes K_C^{\otimes nr}); \\ [V] & \mapsto & [V \otimes K_C^{\otimes n}]; \end{array} \qquad \begin{array}{c} \mathcal{M}_r(L) \to \mathcal{M}_r(L^{\otimes -1} \otimes K_C^{\otimes nr}) \\ [V] \mapsto [V^\vee \otimes K_C^{\otimes n}], \end{array}$$

where $V^\vee$ denotes the dual of V. Next, introduce the union

$$\mathcal{M}_{C,r}^L := \cup_{n\in\mathbb{Z}} \left(\mathcal{M}_r(L \otimes K_C^{\otimes nr}) \cup \mathcal{M}_r(L^{\otimes -1} \otimes K_C^{\otimes nr})\right).$$

With this, clearly, we may and indeed always assume that

$$0 \leq d(L) \leq r(g-1).$$

Furthermore, introduce the partial non-abelian zeta function $\xi_{C,r,\mathbb{F}_q}^L(s)$ by setting

$$\xi_{C,r,\mathbb{F}_q}^L(s) := \sum_{V\in[V]\in\mathcal{M}_{C,r}^L} \frac{q^{h^0(C,V)}-1}{\#\,\mathrm{Aut}(V)} \cdot \left(q^{-s}\right)^{\chi(C,V)}, \qquad \mathrm{Re}(s)>1.$$

Clearly, then

$$\xi_{C,r,\mathbb{F}_q}(s) = \sum_L \xi_{C,r,\mathbb{F}_q}^L(s)$$

where L runs over all line bundles appeared in the following (disjoint) union

$$\cup_{d\in\mathbb{Z}}\mathcal{M}_{C,r}(d) = \cup_{L,0\le d(L)\le r(g-1)}\mathcal{M}_{C,r}^L.$$

Here we reminder the reader that the vanishing result of Lemma 1.2.2.1 has been used.

Therefore, to establish the functional equation for $\xi_{C,r,\mathbb{F}_q}(s)$, it suffices to show that

$$\xi_{C,r,\mathbb{F}_q}^L(s) = \xi_{C,r,\mathbb{F}_q}^L(1-s).$$

For this, we have the following:

THEOREM *For $Re(s)>1$,*

$$\begin{aligned}(*)\quad \xi_{C,r,\mathbb{F}_q}^L(s) = \frac{1}{2}&\sum_{V\in[V]\in\mathcal{M}_{C,r}^L;0\le d(V)\le r(2g-2)} \frac{q^{h^0(C,V)}}{\#\,Aut(V)}\cdot\left[(q^{-s})^{\chi(C,V)}+(q^{s-1})^{\chi(C,V)}\right]\\ &+\left[\frac{q^{(1-s)\cdot(d(L)-r(g-1))}}{q^{(s-1)\cdot r(2g-2)}-1}+\frac{q^{s\cdot(d(L)-r(g-1))}}{q^{(-s)\cdot r(2g-2)}-1}\right.\\ &\quad\left.+\frac{q^{(s-1)\cdot(d(L)-r(g-1))}}{q^{(s-1)\cdot r(2g-2)}-1}+\frac{q^{(-s)\cdot(d(L)-r(g-1))}}{q^{(-s)\cdot r(2g-2)}-1}\right]\cdot\beta_{C,r,\mathbb{F}_q}(L).\end{aligned}$$

Here $\beta_{C,r,\mathbb{F}_q}(L) := \sum_{V\in[V]\in\mathcal{M}_{C,r}(L)}\frac{1}{\#\,Aut(E)}$ denotes the Harder-Narasimhan number. In particular, (a) *$\xi_{C,r,\mathbb{F}_q}^L(s)$ satisfies the functional equation $\xi_{C,r,\mathbb{F}_q}^L(s)=\xi_{C,r,\mathbb{F}_q}^L$* $(1-s)$; (b) *the Harder-Narasimhan number $\beta_{C,r,\mathbb{F}_q}(L)$ is given by the leading term of the singularities of $\xi_{C,r,\mathbb{F}_q}^L(s)$ at $s=0$ and $s=1$.*

Proof. It suffices to prove (*). For this, set

$$I(s) = \sum_{V\in[V]\in\mathcal{M}_{C,r}^L;0\le d(V)\le r(2g-2)} \frac{q^{h^0(C,V)}}{\#\,\mathrm{Aut}(V)}\cdot(q^{-s})^{\chi(C,V)}$$

and

$$II(s) = \sum_{V\in[V]\in\mathcal{M}_{C,r}^{L};=d(V)>r(2g-2)} \frac{q^{h^0(C,V)}}{\#\,\mathrm{Aut}(V)}\cdot(q^{-s})^{\chi(C,V)} - \sum_{V\in[V]\in\mathcal{M}_{C,r}^{L};d(V)\geq 0} \frac{1}{\#\,\mathrm{Aut}(V)}\cdot(q^{-s})^{\chi(C,V)}.$$

Thus,

$$\xi_{C,r,\mathbb{F}_q}^{L}(s) = I(s) + II(s).$$

So it suffices to show the following:

LEMMA. *With the same notation as above:*
(a) $I(s) = \frac{1}{2}\sum_{V\in[V]\in\mathcal{M}_{C,r}^{L};0\leq d(V)\leq r(2g-2)} \frac{q^{h^0(C,V)}}{\#\,\mathrm{Aut}(V)}\cdot\left[(q^{-s})^{\chi(C,V)} + (q^{s-1})^{\chi(C,V)}\right]$;
and
(b)

$$II(s) = \left[\frac{q^{(1-s)\cdot(d(L)-r(g-1))}}{q^{(s-1)\cdot r(2g-2)}-1} + \frac{q^{s\cdot(d(L)-r(g-1))}}{q^{(-s)\cdot r(2g-2)}-1} + \frac{q^{(s-1)\cdot(d(L)-r(g-1))}}{q^{(s-1)\cdot r(2g-2)}-1} + \frac{q^{(-s)\cdot(d(L)-r(g-1))}}{q^{(-s)\cdot r(2g-2)}-1}\right] \cdot\ \beta_{C,r,\mathbb{F}_q}(L).$$

Proof. (a) comes from the Riemann-Roch theorem and Serre duality. Indeed,

$$\begin{aligned}
I(s) &= \frac{1}{2}\left(\sum_{V\in[V]\in\mathcal{M}_{C,r}^{L};0\leq d(E)\leq r(2g-2)} \frac{q^{h^0(C,V)}}{\#\,\mathrm{Aut}(V)}\cdot(q^{-s})^{\chi(C,V)}\right.\\
&\qquad\left. + \sum_{V^\vee\otimes K_C\in\mathcal{M}_{C,r}^{L};0\leq d(V^\vee\otimes K_C)\leq r(2g-2)} \frac{q^{h^0(C,V^\vee\otimes K_C)}}{\#\,\mathrm{Aut}(V^\vee\otimes K_C)}\cdot(q^{-s})^{\chi(C,V^\vee\otimes K_C)}\right)\\
&= \frac{1}{2}\sum_{V\in[V]\in\mathcal{M}_{C,r}^{L};0\leq d(E)\leq r(2g-2)} \left[\frac{q^{h^0(C,V)}}{\#\,\mathrm{Aut}(V)}\cdot(q^{-s})^{\chi(C,V)} + \frac{q^{h^1(C,V^\vee\otimes K_C)}}{\#\,\mathrm{Aut}(V^\vee\otimes K_C)}\right.\\
&\qquad\left.\cdot(q^{1-s})^{\chi(C,V^\vee\otimes K_C)} \sum_{V\in[V]\in}\right]\\
&= \frac{1}{2}\sum_{V\in[V]\in\mathcal{M}_{C,r}^{L};0\leq d(V)\leq r(2g-2)} \frac{q^{h^0(C,V)}}{\#\,\mathrm{Aut}(V)}\cdot\left[(q^{-s})^{\chi(C,V)} + (q^{s-1})^{\chi(C,V)}\right].
\end{aligned}$$

As for (b), clearly by the vanishing result,

$$\begin{aligned}
T_{r,L}(s) &= \sum_{V\in[V]\in\mathcal{M}_{C,r}^{L};d(E)>r(2g-2)} \frac{1}{\#\ \mathrm{Aut}(E)}\cdot(q^{1-s})^{\chi(C,V)} \\
&\quad - \sum_{V\in[V]\in\mathcal{M}_{C,r}^{L};d(E)\geq 0} \frac{1}{\#\mathrm{Aut}(E)}\cdot(q^{-s})^{\chi(C,V)} \\
&= \Bigg(\sum_{V\in[V]\in\mathcal{M}_{C,r}(L\otimes K_C^{\otimes rn});d(L)+rn(2g-2)>r(2g-2)} \frac{1}{\#\mathrm{Aut}(V)}\cdot(q^{1-s})^{\chi(C,E)} \\
&\qquad - \sum_{V\in[V]\in\mathcal{M}_{C,r}(L^{-1}\otimes K_C^{\otimes rn});-d(L)+rn(2g-2)\geq 0} \frac{1}{\#\mathrm{Aut}(V)}\cdot(q^{-s})^{\chi(C,E)}\Bigg) \\
&\quad + \Bigg(\sum_{V\in[V]\in\mathcal{M}_{C,r}(L^{-1}\otimes K_C^{\otimes rn});-d(L)+rn(2g-2)>r(2g-2)} \frac{1}{\#\mathrm{Aut}(V)}\cdot(q^{1-s})^{\chi(C,E)} \\
&\qquad - \sum_{V\in[V]\in\mathcal{M}_{C,r}(L\otimes K_C^{\otimes rn});d(L)+rn(2g-2)>0} \frac{1}{\#\mathrm{Aut}(V)}\cdot(q^{-s})^{\chi(C,E)}\Bigg).
\end{aligned}$$

But $\chi(C,V)$ depends only on $d(V)$. Thus, accordingly,

$$\begin{aligned}
II(s) &= \Bigg[\Bigg(\sum_{n=1}^{\infty}(q^{1-s})^{d(L)+nr(2g-2)-r(g-1)} - \sum_{n=1}^{\infty}(q^{-s})^{-d(L)+nr(2g-2)-r(g-1)}\Bigg) \\
&\quad + \Bigg(\sum_{n=2}^{\infty}(q^{1-s})^{-d(L)+nr(2g-2)-r(g-1)} - \sum_{n=0}^{\infty}(q^{-s})^{d(L)+nr(2g-2)-r(g-1)}\Bigg)\Bigg]\cdot\beta_{C,r}(L) \\
&= \Bigg[\frac{q^{(1-s)\cdot(d(L)-r(g-1))}}{q^{(s-1)\cdot r(2g-2)}-1} + \frac{q^{s\cdot(d(L)-r(g-1))}}{q^{(-s)\cdot r(2g-2)}-1} + \frac{q^{(s-1)\cdot(d(L)-r(g-1))}}{q^{(s-1)\cdot r(2g-2)}-1} + \frac{q^{(-s)\cdot(d(L)-r(g-1))}}{q^{(-s)\cdot r(2g-2)}-1}\Bigg] \\
&\quad \cdot\beta_{C,r,\mathbb{F}_q}(L).
\end{aligned}$$

This completes the proof of the lemma, and hence the Theorem and the Functional Equation for rank r zeta functions.

I.2. Global non-abelian zeta functions for curves. In this section, we introduce new non-abelian zeta functions for curves defined over number fields via the Euler product formalism, based on our study of non-abelian zetas for curves defined over finite fields in the previous section. Our main result here is about a convergence region of such a Euler product. Key ingredients of our proof are a result of (Harder-Narasimhan) Siegel, an ugly yet very precise formula for our local zeta functions, the Clifford Lemma for semi-stable vector bundles, and Weil's theorem on the Riemann Hypothesis for Artin zeta functions.

2.1. Preparations.

2.1.1. Invariants α, β and γ. Let C be a regular, reduced, irreducible projective curve of genus g defined over the finite field $\mathbb{F}_q$ with q elements. As in I.1, we then get (the subset of $\mathbb{F}_q$-rational points of) the associated moduli spaces $\mathcal{M}_{E,r}(L)$ and $\mathcal{M}_{C,r}(d)$. Recall that in I.1, motivated by a work of Harder-Narasimhan [HN], we, following Desale-Ramanan [DR], defined the Harder-Narasimhan numbers $\beta_{C,r,\mathbb{F}_q}(L), \beta_{C,r,\mathbb{F}_q}(d)$, which are very useful in the discussion of our zeta functions. Now we introduce new invariants for C by setting

$$\alpha_{C,r,\mathbb{F}_q}(d) := \sum_{V\in[V]\in\mathcal{M}_{C,r}(d)(\mathbb{F}_q)} \frac{q^{h^0(C,V)}}{\#\mathrm{Aut}(V)}, \qquad \gamma_{C,r,\mathbb{F}_q}(d) := \sum_{V\in[V]\in\mathcal{M}_{C,r}(d)(\mathbb{F}_q)} \frac{q^{h^0(C,V)}-1}{\#\mathrm{Aut}(V)},$$

and similarly define $\alpha_{C,r,\mathbb{F}_q}(L)$ and $\gamma_{C,r,\mathbb{F}_q}(L)$.

LEMMA. *With the same notation as above:*
(a) *for* $\alpha_{C,r,\mathbb{F}_q}(d)$,

$$\alpha_{C,r,\mathbb{F}_q}(d) = \begin{cases} \beta_{C,r,\mathbb{F}_q}(d); & d < 0;\\ \alpha_{C,r,\mathbb{F}_q}(r(2g-2)-d)\cdot q^{d-r(g-1)}, & 0 \le d \le r(2g-2);\\ \beta_{C,r,\mathbb{F}_q}(d)\cdot q^{d-r(g-1)}, & d > r(2g-2); \end{cases}$$

(b) *for* $\beta_{C,r,\mathbb{F}_q}(d)$,

$$\beta_{C,r,\mathbb{F}_q}(\pm d + rn) = \beta_{C,r,\mathbb{F}_q}(d) \qquad n \in \mathbb{Z};$$

(c) *for* $\gamma_{C,r,\mathbb{F}_q}(d)$,

$$\gamma_{C,r,\mathbb{F}_q}(d) = \alpha_{C,r,\mathbb{F}_q}(d) - \beta_{C,r,\mathbb{F}_q}(d).$$

Proof. (c) simply comes from the definition, while (b) is a direct consequence of Lemma 2 in 1.2.2 and the fact that $\mathrm{Aut}(V) \simeq \mathrm{Aut}(V^\vee)$ for a vector bundle V. So it suffices to prove (a).

When $d < 0$, the relation is deduced from the fact that $h^0(C,V) = 0$ if V is a semi-stable vector bundle with strictly negative degree; when $0 \le d \le r(2g-2)$, the result comes from the Riemann-Roch and Serre duality; finally when $d > r(2g-2)$, the result is a direct consequence of the Riemann-Roch and the fact that $h^1(C,V) = 0$ if V is a semi-stable vector bundle with degree strictly bigger than $r(2g-2)$.

We here remind the reader that this lemma and Lemma 2 in 1.2.2 tell us that all $\alpha_{C,r,\mathbb{F}_q}(d), \beta_{C,r,\mathbb{F}_q}(d)$ and $\gamma_{C,r,\mathbb{F}_q}(d)$'s for all $d \in \mathbb{Z}$ may be calculated from a finite subset of them, that is, from $\alpha_{C,r,\mathbb{F}_q}(i), \beta_{C,r,\mathbb{F}_q}(j)$ with $i = 0, \dots, r(g-1)$ and $j = 0, \dots, r-1$.

2.1.2. Asymptotic behaviors of α, β and γ. For later use, we here discuss the asymptotic behavior of $\alpha_{C,r,\mathbb{F}_q}(d)$, $\beta_{C,r,\mathbb{F}_q}(d)$, and $\gamma_{C,r,\mathbb{F}_q}(0)$ when $q \to \infty$.

PROPOSITION. *With the same notation as above, when $q \to \infty$:*
(a) *For all d,*

$$\beta_{C,r,\mathbb{F}_q}(d) = O\left(q^{r^2(g-1)}\right);$$

(b)

$$\frac{q^{(r-1)(g-1)}}{\gamma_{C,r,\mathbb{F}_q}(0)} = O(1).$$

(c) *For $0 \le d \le r(g-1)$,*

$$\frac{\alpha_{C,r,\mathbb{F}_q}(d)}{q^{d/2+r+r^2(g-1)}} = O(1).$$

Proof. Following Harder and Narasimhan [HN], a result of Siegel on quadratic forms which is equivalent to the fact that Tamagawa number of SL_r is 1, may be understood via the following relation on automorphism groups of rank r vector bundles:

$$\sum_{V:r(V)=r,\det(V)=L} \frac{1}{\#\mathrm{Aut}(V)} = \frac{q^{(r^2-1)(g-1)}}{q-1} \cdot \zeta_C(2)\dots\zeta_C(r).$$

Here V runs over all rank r vector bundles with determinant L and $\zeta_C(s)$ denotes the Artin zeta function of C. Thus,

$$0 < \beta_{C,r,\mathbb{F}_q}(L) \le \frac{q^{(r^2-1)(g-1)}}{q-1} \cdot \zeta_C(2)\dots\zeta_C(r).$$

This implies

$$\begin{aligned}\beta_{C,r,\mathbb{F}_q}(d) &= \prod_{i=1}^{2g}(1-\omega_{C,1,\mathbb{F}_q}(i)) \cdot \beta_{C,r,\mathbb{F}_q}(L) \\ &\le \prod_{i=1}^{2g}(1-\omega_{C,1,\mathbb{F}_q}(i)) \cdot \frac{q^{(r^2-1)(g-1)}}{q-1} \cdot \zeta_C(2)\dots\zeta_C(r).\end{aligned}$$

Here two facts are used:

(1) The number of $\mathbb{F}_q$-rational points of degree d Jacobian $J^d(C)$ is equal to $\prod_{i=1}^{2g}(1-\omega_{C,1,\mathbb{F}_q}(i))$; and

(2) a result of Desale and Ramanan, which says that for any two $L, L' \in \mathrm{Pic}^d(C)$, $\beta_{C,r,\mathbb{F}_q}(L) = \beta_{C,r,\mathbb{F}_q}(L')$. (See e.g., [DR, Prop 1.7.(i)])

Thus by Weil's theorem on Riemann Hypothesis on Artin zeta functions [W1],

$$|\omega_{C,1,\mathbb{F}_q}(i)| = O(q^{1/2}), \qquad i = 0, \ldots, 2g.$$

This then completes the proof of (a).

To prove (b), first note that (b) is equivalent to that, asymptotically, the lower bound of $\gamma_{C,r,\mathbb{F}_q}(0)$ is at least $q^{(r-1)(g-1)}$. To show this, note that

$$\begin{aligned}\gamma_{C,r,\mathbb{F}_q}(0) &\geq \sum_{V=\mathcal{O}_C\oplus L_2\oplus\ldots\oplus L_r, L_2,\ldots,L_r\in\mathrm{Pic}^0(C), \#\{\mathcal{O}_C,L_2,\ldots,L_r\}=r} \frac{q^{h^0(C,V)}-1}{\#\,\mathrm{Aut}(V)}\\ &= \frac{1}{(q-1)^{r-1}} \sum_{V=\mathcal{O}_C\oplus L_2\oplus\ldots\oplus L_r, L_2,\ldots,L_r\in\mathrm{Pic}^0(C), \#\{\mathcal{O}_C,L_2,\ldots,L_r\}=r} 1.\end{aligned}$$

Now, by the above cited result of Weil again, as $q \to \infty$,

$$\sum_{V=\mathcal{O}_C\oplus L_2\oplus\ldots\oplus L_r, L_2,\ldots,L_r\in\mathrm{Pic}^0(C), \#\{\mathcal{O}_C,L_2,\ldots,L_r\}=r} 1 = O(q^{g(r-1)}).$$

So we have (b) as well.

Just as in (a), (c) is about to give an upper bound for $\alpha_{C,r,\mathbb{F}_q}(d)$ for $0 \leq d \leq r(2g-2)$. For this, we first recall the following:

Clifford Lemma. (See e.g., [B-PBGN, Theorem 2.1].) *Let V be a semi-stable bundle of rank r and degree d with $0 \leq \mu(V) \leq 2g-2$. Then*

$$h^0(C, V) \leq r + \frac{d}{2}.$$

Thus,

$$\alpha_{C,r,\mathbb{F}_q}(d) \leq q^{\frac{d}{2}+r} \cdot \beta_{C,r,\mathbb{F}_q}(d).$$

With this, (c) is a direct consequence of (a).

2.1.3. Ugly formula. Recall that the rationality of $\zeta_{C,r,\mathbb{F}_q}(s)$ says that there exists a degree $2rg$ polynomial $P_{C,r,\mathbb{F}_q}(t) \in \mathbb{Q}[t]$ such that

$$Z_{C,r,\mathbb{F}_q}(t) = \frac{P_{C,r,\mathbb{F}_q}(t)}{(1-t^r)(1-q^r t^r)}.$$

Thus we may set

$$P_{C,r,\mathbb{F}_q}(t) =: \sum_{i=0}^{2rg} a_{C,r,\mathbb{F}_q}(i) t^i.$$

On the other hand, by the functional equation for $\xi_{C,r,\mathbb{F}_q}(t)(s)$, we have

$$P_{C,r,\mathbb{F}_q}(t) = P_{C,r,\mathbb{F}_q}\left(\frac{1}{qt}\right) \cdot q^{rg} \cdot t^{2rg}.$$

Hence, by comparing coefficients on both sides, we get the following:

LEMMA. *With the same notation as above, for* $i = 0, 1, \ldots, rg-1$,

$$a_{C,r,\mathbb{F}_q}(2rg - i) = a_{C,r,\mathbb{F}_q}(i) \cdot q^{rg-i}.$$

Now, to determine $P_{C,r,\mathbb{F}_q}(t)$ and hence $\zeta_{C,r,\mathbb{F}_q}(s)$ it suffices to find $a_{C,r,\mathbb{F}_q}(i)$ for $i = 0, 1, \ldots, rg$.

PROPOSITION. (An Ugly Formula) *With the same notation as above,*

$$a_{C,r,\mathbb{F}_q}(i) = \begin{cases} \alpha_{C,r,\mathbb{F}_q}(d) - \beta_{C,r,\mathbb{F}_q}(d), & 0 \le i \le r-1; \\ \alpha_{C,r,\mathbb{F}_q}(d) - (q^r+1)\alpha_{C,r,\mathbb{F}_q}(d-r) & \\ \quad + q^r \beta_{C,r,\mathbb{F}_q}(d-r), & r \le i \le 2r-1; \\ \alpha_{C,r,\mathbb{F}_q}(d) - (q^r+1)\alpha_{C,r,\mathbb{F}_q}(d-r) & \\ \quad + q^r \alpha_{C,r,\mathbb{F}_q}(d-2r), & 2r \le i \le r(g-1)-1; \\ -(q^r+1)\alpha_{C,r,\mathbb{F}_q}(r(g-2)) & \\ \quad + q^r \alpha_{C,r,\mathbb{F}_q}(r(g-3)) & \\ \quad + \alpha_{C,r,\mathbb{F}_q}(r(g-1)), & i = r(g-1); \\ \alpha_{C,r,\mathbb{F}_q}(d) - (q^r+1)\alpha_{C,r,\mathbb{F}_q}(d-r) & \\ \quad + \alpha_{C,r,\mathbb{F}_q}(d-2r) q^r, & r(g-1)+1 \le i \le rg-1; \\ 2q^r \alpha_{C,r,\mathbb{F}_q}(r(g-2)) & \\ \quad - (q^r+1)\alpha_{C,r,\mathbb{F}_q}(r(g-1)), & i = rg. \end{cases}$$

Proof. By definition,

$$
\begin{aligned}
&Z_{C,r,\mathbb{F}_q}(t)\\
&\quad=\left(\sum_{d=0}^{r(2g-2)}+\sum_{d=r(2g-2)+1}^{\infty}\right)\sum_{V\in[V]\in\mathcal{M}_{C,r}(d),d\geq 0}\frac{q^{h^0(C,V)}-1}{\#\mathrm{Aut}(V)}t^d\\
&\quad=\sum_{d=0}^{r(2g-2)}\sum_{V\in[V]\in\mathcal{M}_{C,r}(d)}\frac{q^{h^0(C,V)}-1}{\#\mathrm{Aut}(V)}t^d\\
&\qquad+\sum_{i=1}^{r}\sum_{n=2g-2}^{\infty}\sum_{d=rn+i}\sum_{V\in[V]\in\mathcal{M}_{C,r}(d)}\frac{q^{rn+i-r(g-1)}-1}{\#\mathrm{Aut}(V)}t^{rn+i}\\
&\quad=\sum_{d=0}^{r(2g-2)}\sum_{V\in[V]\in\mathcal{M}_{C,r}(d)(\mathbb{F}_q)}\frac{q^{h^0(C,V)}-1}{\#\mathrm{Aut}(V)}t^d\\
&\qquad+\frac{q^{r(1-g)}}{1-(qt)^r}(qt)^{r(2g-2)}\sum_{i=1}^{r}\beta_{C,r,\mathbb{F}_q}(i)(qt)^i-\frac{1}{1-t^r}t^{r(2g-2)}\sum_{i=1}^{r}\beta_{C,r,\mathbb{F}_q}(i)t^i,
\end{aligned}
$$

by a similar calculation as in the proof of Lemma 1.2.4.(b). Now

$$
\sum_{d=0}^{r(2g-2)}=\sum_{d=0,r(2g-2)}+\sum_{d=1,r(2g-2)-1}+\cdots+\sum_{d=r(g-1)-1,r(g-1)+1}+\sum_{d=r(g-1)}.
$$

Thus, by Riemann-Roch, Serre duality and Lemma 2.1.1, we conclude that

$$
\begin{aligned}
&\sum_{d=0}^{r(2g-2)}\sum_{V\in[V]\in\mathcal{M}_{C,r}(d)}\frac{q^{h^0(C,V)}-1}{\#\mathrm{Aut}(V)}t^d\\
&\quad=\sum_{d=0}^{r(g-1)-1}\left[\alpha_{C,r,\mathbb{F}_q}(d)\left(t^d+q^{r(g-1)-d}t^{r(2g-2)-d}\right)-\beta_{C,r,\mathbb{F}_q}(d)\left(t^d+t^{r(2g-2)-d}\right)\right]\\
&\qquad+\left(\alpha_{C,r,\mathbb{F}_q}(r(g-1))-\beta_{C,r,\mathbb{F}_q}(r(g-1))\right)\cdot t^{r(g-1)}.
\end{aligned}
$$

With all this, together with Lemma 2 in 1.2.2 and the Lemma in 2.1.1, and by a couple of pages of routine calculations, we are led to the ugly yet very precise formula in the proposition.

2.2. Global non-abelian zeta functions for curves.

2.2.1. Definition. Let $\mathcal{C}$ be a regular, reduced, irreducible projective curve of genus g defined over a number field F. Let S_{bad} be the collection of all infinite places and those finite places of F at which $\mathcal{C}$ does not have good reductions. As usual, a place v of F is called good if $v\notin S_{\text{bad}}$.

Thus, in particular, for any good place v of F, the v-reduction of $\mathcal{C}$, denoted as $\mathcal{C}_v$, gives a regular, reduced, irreducible projective curve defined over the residue field $F(v)$ of F at v. Denote the cardinal number of $F(v)$ by q_v. Then, by the construction of I.1, we obtain associated rank r non-abelian zeta function $\zeta_{\mathcal{C}_v,r,\mathbb{F}_{q_v}}(s)$. Moreover, from the rationality of $\zeta_{\mathcal{C}_v,r,\mathbb{F}_{q_v}}(s)$, there exists a degree $2rg$ polynomial $P_{\mathcal{C}_v,r,\mathbb{F}_{q_v}}(t) \in \mathbb{Q}[t]$ such that

$$Z_{\mathcal{C}_v,r,\mathbb{F}_{q_v}}(t) = \frac{P_{\mathcal{C}_v,r,\mathbb{F}_{q_v}}(t)}{(1-t^r)(1-q^r t^r)}.$$

Clearly,

$$P_{\mathcal{C}_v,r,\mathbb{F}_{q_v}}(0) = \gamma_{\mathcal{C}_v,r,\mathbb{F}_{q_v}}(0) \neq 0.$$

Thus it makes sense to introduce the polynomial $\tilde{P}_{\mathcal{C}_v,r,\mathbb{F}_{q_v}}(t)$ with constant term 1 by setting

$$\tilde{P}_{\mathcal{C}_v,r,F(v)}(t) := \frac{P_{\mathcal{C}_v,r,F(v)}(t)}{P_{\mathcal{C}_v,r,F(v)}(0)}.$$

Now by definition, *the rank r non-abelian zeta function $\zeta_{\mathcal{C},r,F}(s)$ of $\mathcal{C}$* over F is the following Euler product

$$\zeta_{\mathcal{C},r,F}(s) := \prod_{v:\text{good}} \frac{1}{\tilde{P}_{\mathcal{C}_v,r,\mathbb{F}_{q_v}}(q_v^{-s})}, \qquad \mathrm{Re}(s) \gg 0.$$

Clearly, when $r = 1$, $\zeta_{\mathcal{C},r,F}(s)$ coincides with the classical Hasse-Weil zeta function for C over F [H].

2.2.2. Convergence. At this earlier stage of the study of our non-abelian zeta functions, the central problem is to justify the above definition. That is to say, to show the above Euler product converges. In this direction, we have the following:

THEOREM. *Let $\mathcal{C}$ be a regular, reduced, irreducible projective curve defined over a number field F. Then its associated rank r global non-abelian zeta function $\zeta_{\mathcal{C},r,F}(s)$ converges when $Re(s) \geq 1 + g + (r^2 - r)(g-1)$.*

Proof. Clearly, it suffices to show that for the reciprocal roots $\omega_{C,r,\mathbb{F}_q}(i), i = 1, \ldots, 2rg$, of $P_{C,r,\mathbb{F}_q}(t)$ associated to curves C over finite fields $\mathbb{F}_q$,

$$|\omega_{C,r,\mathbb{F}_q}(i)| = O(q^{g+(r^2-1)(g-1)}).$$

Thus we are led to estimate coefficients of $P_{C,r,\mathbb{F}_q}(t)$. Since we have the ugly yet very precise formula for these coefficients, i.e., the lemma and the proposition in

2.1.3, it suffices to give upper bounds for $\alpha_{C,r,\mathbb{F}_q}(i), \beta_{C,r,\mathbb{F}_q}(j)$ and a lower bound for $\gamma_{C,f,\mathbb{F}_q}(0)$, the constant term of $P_{C,r,\mathcal{F}_q}(t)$. Thus, to complete the proof, we only need to cite the Proposition in 2.1.2.

Question. For any regular, reduced, irreducible projective curve $\mathcal{C}$ of genus g defined over a number field F, does its associated rank r global non-abelian zeta function $\zeta_{\mathcal{C},r,F}(s)$ admit meromorphic continuation to the whole complex s-plane?

Recall that even when $r = 1$, i.e., for the classical Hasse-Weil zeta functions, this is still quite open.

2.2.3. Working hypothesis. As in the theory of abelian zeta functions, we want to use our non-abelian zeta functions to study non-abelian arithmetic aspect of curves. Motivated by the classical analytic class number formula for Dedekind zeta functions and its counterpart BSD conjecture for Hasse-Weil zeta functions of elliptic curves, we expect that our non-abelian zeta function can be used to understand the Weil-Petersson volumes of moduli space of stable bundles as well as the associated Tamagawa measures.

As such, local factors for “bad” places are needed. Our suggestion is as follows: for Γ-factors, we take those coming from the functional equation for $\zeta_F(rs) \cdot \zeta_F(r(s-1))$, where $\zeta_F(s)$ denotes the standard Dedekind zeta function of F; while for finite bad places, we first use the semi-stable reduction for curves to find a semi-stable model for $\mathcal{C}$, then use Seshadri’s moduli spaces of parabolic bundles to construct polynomials for singular fibers, which usually have degree lower than $2rg$. With all this done, we then can introduce the so-called completed rank r non-abelian zeta function for $\mathcal{C}$ over F, or better, the completed rank r non-abelian zeta function $\xi_{X,r,\mathcal{O}_F}(s)$ for a semi-stable model $X \to \mathrm{Spec}(\mathcal{O}_F)$ of $\mathcal{C}$. Here $\mathcal{O}_F$ denotes the ring of integers of F. (If necessary, we take a finite extension of F.)

Question. Does the meromorphic extension of $\xi_{X,r,\mathcal{O}_F}(s)$, if it exists, satisfy the functional equation

$$\xi_{X,r,\mathcal{O}_F}(s) = \pm\, \xi_{X,r,\mathcal{O}_F}\left(1 + \frac{1}{r} - s\right)?$$

Remark. From our study for non-abelian zeta functions of elliptic curves [We3], we obtain the following “absolute Euler product” for rank 2 zeta functions of elliptic curves

$$\zeta_2(s) \;=\; \prod_{p>2;\mathrm{prime}} \frac{1}{1+(p-1)p^{-s}+(2p-4)p^{-2s}+(p^2-p)p^{-3s}+p^2p^{-4s}}$$

$$= \prod_{p>2;\mathrm{prime}} \frac{1}{A_p(s) + B_p(s)p^{-2s}}, \qquad \mathrm{Re}(s) > 2$$

with

$$A_p(s) = 1 + (p-1)p^{-s} + (p-2)p^{-2s}, \qquad B_p(s) = (p-2) + (p^2 - p)p^{-s} + p^2 p^{-2s}.$$

Set $t := q^{-s}$ and $a_p(t) := A_p(s), b_p(t) := B_p(s)$. Then in $\mathbb{Z}[t]$, we have the factorization

$$a_p(t) = (1 + (p-2)t)(1+t), \qquad b_p(t) = ((p-2) + pt)(1 + pt)$$

and

$$a_p\left(\frac{1}{pt}\right) = \frac{1}{p^2t^2} \cdot b_p(t).$$

As pointed out to me by Kohnen,

$$1 + (p-1)p^{-s} + (2p-4)p^{-2s} + (p^2 - p)p^{-3s} + p^2 p^{-4s}$$

is quite similar to Andrianov's genus two spinor L-function. (See e.g. [We1].)

I.3. Non-abelian zeta functions and infinitesimal structures of Brill-Noether loci. In this section, we study the infinitesimal structures of the so-called non-abelian Brill-Noether loci for rank two semi-stable vector bundles over genus two curves. As an application, we calculate the corresponding rank two non-abelian zeta functions. During this process, we see clearly how Weierstrass points, intrinsic arithmetic invariants of curves [We1], contribute to our zeta functions among others.

In this section we assume that the characteristic of the base field is strictly bigger than 2 for simplicity.

3.1. Invariants $\beta_{C,2,\mathbb{F}_q}(d)$. Let C be a genus two regular reduced irreducible projective curve defined over $\mathbb{F}_q$. Here we want to calculate Harder-Narasimhan numbers $\beta_{C,2,\mathbb{F}_q}(d)$ for all d. Note that from the lemma in 2.1.1,

$$\beta_{C,2,\mathbb{F}_q}(d) = \beta_{C,2,\mathbb{F}_q}(d + 2n).$$

So it suffices to calculate $\beta_{C,2,\mathbb{F}_q}(d)$ when $d = 0, 1$. For this, we cite the following result of Desale and Ramanan:

PROPOSITION. [DR] *With the same notation as above, for $L \in Pic^d(C), d = 0, 1$,*

$$\beta_{C,2,\mathbb{F}_q}(L) = \frac{q^3}{q-1} \cdot \zeta_C(2) - q \prod_{i=1}^4 (1-\omega_i) \cdot \sum_{d_1+d_2=d, d_1>d_2} \frac{\beta_{C,1,\mathbb{F}_q}(d_1)\beta_{C,1,\mathbb{F}_q}(d_2)}{q^{d_1-d_2}}.$$

Here $\zeta_C(s)$ denotes the Artin zeta function for C and $\omega_1, \ldots, \omega_4$ are the roots of the associated Z-function $Z_C(s)$, i.e., $\omega_{C,1,\mathbb{F}_q}(i), i = 0, \ldots, 4 = 2 \times 2$ in our notation.

Thus, in particular, $\beta_{C,2,\mathbb{F}_q}(L)$ is independent of L.

LEMMA. *With the same notation as above, for $d = 0, 1$*

$$\beta_{C,2,\mathbb{F}_q}(d) = \frac{q^3}{q-1} \cdot \zeta_C(2) \cdot \prod_{i=1}^4 (1-\omega_i) - \frac{q^{d+1}}{(q-1)^2(q^2-1)} \cdot \prod_{i=1}^4 (1-\omega_i)^4.$$

Proof. This comes from the following two facts:
(1) for all d,

$$\beta_{C,1,\mathbb{F}_q}(d) = \frac{\prod_{i=1}^4 (1-\omega_i)}{q-1};$$

(2) the number of $\mathbb{F}_q$-rational points of $\mathrm{Pic}^d(C)$ is equal to $\prod_{i=1}^4 (1-\omega_i)$.

3.2. Invariants α & γ: Easy parts.

3.2.1. Infinitesimal structures: a taste. Here we want to calculate $\alpha_{C,2,\mathbb{F}_q}(0)$. By the Lemma in 3.1, it suffices to give $\gamma_{C,2,\mathbb{F}_q}(0)$. So we are lead to study $\gamma_{C,2,\mathbb{F}_q}(L)$ which is supported over the Brill-Noether locus

$$W_{C,2}^0(L) := \{[V] \in \mathcal{M}_{C,2}(L) : h^0(C, \mathrm{Gr}(V)) \geq 1\}.$$

(In general, as in [B-PGN], we define the Brill-Noether locus by

$$W_{C,2}^k(L) := \{[V] \in \mathcal{M}_{C,2}(L) : h^0(C, \mathrm{Gr}(V)) \geq k+1\}.)$$

Note that no degree zero stable bundle admits nontrivial global sections, so $W_{C,2}^0(L) := \{[\mathcal{O}_C \oplus L]\}$ consists of only one single point.

(a) If $L = \mathcal{O}_C$, then $W_{C,2}^0(\mathcal{O}_C) = W_{C,2}^1(\mathcal{O}_C)$. Moreover, infinitesimally, $V = \mathcal{O}_C \oplus \mathcal{O}_C$ or V corresponds to all nontrivial extensions

$$0 \to \mathcal{O}_C \to V \to \mathcal{O}_C \to 0$$

which are parametrized by $\mathbb{P}\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) \simeq \mathbb{P}^1$. Thus, by definition,

$$\gamma_{C,2,\mathbb{F}_q}(\mathcal{O}_C) = \frac{q^2-1}{(q^2-1)(q^2-q)} + (q+1) \cdot \frac{q-1}{q(q-1)} = \frac{q}{q-1}.$$

(b) If $L \neq \mathcal{O}_C$, then, infinitesimally, $V = \mathcal{O}_C \oplus L$ or V corresponds to the single nontrivial extension

$$0 \to \mathcal{O}_C \to V \to L \to 0.$$

Thus, by definition,

$$\gamma_{C,2,\mathbb{F}_q}(L) = \frac{q-1}{(q-1)^2} + \frac{q-1}{q-1} = \frac{q}{q-1}.$$

Thus we have the following:

LEMMA. *With the same notation as above, for all $L \in Pic^0(C)$,*

$$\gamma_{C,2,\mathbb{F}_q}(L) = \frac{q}{q-1}.$$

In particular,

$$\gamma_{C,2,\mathbb{F}_q}(0) = \frac{q}{q-1} \cdot \prod_{i=1}^{4} (1-\omega_i).$$

3.2.2. Invariants $\alpha_{C,2,\mathbb{F}_q}(1)$. As before, it suffices to calculate $\gamma_{C,2,\mathbb{F}_q}(L)$ for all $L \in \mathrm{Pic}^1(C)$. Note that in this case, all bundles are stable, so $\mathrm{Aut}(V) \simeq \mathbb{F}_q^*$ and

$$W_{C,2}^0(L) \simeq \{V : \text{stable}, r(V) = 2, \det(V) = L, h^0(C, V) \geq 1\}.$$

Moreover, by [B-PGN, Prop. 3.1],

$$W_{C,2}^0(L) = \{V : \text{stable}, r(V) = 2,\ \det(V) = L, h^0(C, V) = 1\}$$

and any $V \in W_{C,2}^0(L)$ admits a nontrivial extension

$$0 \to \mathcal{O}_C \to V \to L \to 0.$$

On the other hand, any nontrivial extension

$$0 \to \mathcal{O}_C \to V \to L \to 0$$

gives rise to a stable bundle. So in fact

$$W^0_{C,2}(L) \simeq \mathbb{P}\mathrm{Ext}^1(L, \mathcal{O}_C) \simeq \mathbb{P}^1.$$

Thus we have the following:

LEMMA. *With the same notation as above, for $L \in Pic^1(C)$,*

$$W^0_{C,2}(L) \simeq \mathbb{P}^1, \qquad \textit{and} \qquad \gamma_{C,2,\mathbb{F}_q}(L) = q + 1.$$

In particular,

$$\gamma_{C,2\mathbb{F}_q}(1) = (q+1) \cdot \prod_{i=1}^{4} (1 - \omega_i).$$

3.3. Infinitesimal structures of non-abelian Brill-Noether loci. We next calculate $\gamma_{C,r,\mathbb{F}_q}(2)$. In general, the level $r(g-1)$, which in our present case corresponds to 2, is the most complicated one. So the discussion here is rather involved.

Let us start with the structures of the non-abelian Brill-Noether loci $W^0_{C,2}(L)$ and $W^1_{C,2}(L)$ for $L \in \mathrm{Pic}^2(C)$. For this, recall the structure map $\pi : C \times C/S_2 \to \mathrm{Pic}^2(C)$ defined by $[(x, y)] \mapsto \mathcal{O}_C(x+y)$. Here S_2 denotes the symmetric group of two symbols which acts naturally on $C \times C$ via $(x, y) \mapsto (y, x)$. One checks that π is a one point blowing-up centered at the canonical line bundle K_C of C. For later use, denote by Δ the image of the diagonal of $C \times C$ in $\mathrm{Pic}^2(C)$.

Next, we want to understand the structure of sublocus $W^0_{C,2}(L)^{\mathrm{ss}}$ of $W^0_{C,2}(L)$ consisting of nonstable but semi-stable vector bundles. By definition, for any $V \in [V] \in W^0_{C,2}(L)^{\mathrm{ss}}$, $\mathrm{Gr}(V) = \mathcal{O}_C(P) \oplus L(-P)$ for a suitable ($\mathbb{F}_q$-rational) point $P \in C$. Thus accordingly:

(a) if $L \neq K_C$, then $W^0_{C,2}(L)^{\mathrm{ss}}$ is parametrized by ($\mathbb{F}_q$-rational points of) C, due to the fact that now $h^0(C, L) = 1$. Write also $L = \mathcal{O}_C(A + B)$ with two points A, B of C, which are unique from the above discussion on the map π, we then conclude that

$$W^1_{C,2}(L) = \{[\mathcal{O}_C(A) \oplus \mathcal{O}_C(B)]\}.$$

(b) if $L = K_C$, then for any P, $K_C = \mathcal{O}_C(P + \iota(P))$ where $\iota : C \to C$ denotes the canonical involution on C. So

$$W^0_{C,2}(L)^{\mathrm{ss}} = \{[\mathcal{O}_C(P) \oplus \mathcal{O}_C(\iota(P))] : P \in C\}.$$

Therefore $W^0_{C,2}(L)^{\mathrm{SS}}$ is parametrized by $\mathbb{P}^1$. Moreover,

$$W^1_{C,2}(K_C) = W^0_{C,2}(L)^{\mathrm{SS}} = \{[\mathcal{O}_C(P) \oplus \mathcal{O}_C(\iota(P))] : P \in C\}.$$

On the other hand, it is easy to check that every nontrivial extension

$$0 \to \mathcal{O}_V \to W \to L \to 0$$

gives rise to a semi-stable vector bundle W, and if W is not stable, then there exists a point $Q \in C$ such that W may also be given by the nontrivial extension

$$0 \to \mathcal{O}_C(Q) \to W \to L(-Q) \to 0.$$

Note also that the kernel of the natural map $H^1(C, \mathrm{Hom}(L, \mathcal{O}_C)) \to H^1(C, \mathrm{Hom}(L(-Q), \mathcal{O}_C))$ is one dimensional. So among all nontrivial extensions $0 \to \mathcal{O}_C \to V \to L \to 0$, which are parametrized by $\mathbb{P}\ \mathrm{Ext}^1(L, \mathcal{O}_C) \simeq \mathbb{P}^2$, the nonstable (yet semi-stable) vector bundles are parametrized by ($\mathbb{F}_q$-rational points of) C when $L \neq K_C$ by (a), or $\mathbb{P}^1$ when $L = K_C$ by (b). (See [NR, Lemma 3.1].) In this way, we have proved the following result on non-abelian Brill-Noether loci for moduli space of $\mathcal{M}_{C,2}(L)$ with L a degree 2 line bundle on a genus two curve, which is not covered by [B-PGN]:

LEMMA. *With the same notation as above, $W^0_{C,2}(L) \simeq \mathbb{P}Ext^1(L, \mathcal{O}_C) \simeq \mathbb{P}^2$, in which the locus $W^0_{C,2}(L)^{SS}$ of semi-stable but not stable bundles is parametrized by C or $\mathbb{P}^1$ according to $L \neq K_C$ or $L = K_C$. More precisely:*

(a) *if $L = \mathcal{O}_C(A + B) \neq K_C$ with A, B two points of C, then $W^0_{C,2}(L)^{SS}$, as a birational image of C under the complete linear system $K_C(A + B)$, is a degree 4 plane curve with a single node located at $W^1_{C,2}(L) = \{[\mathcal{O}_C(A) \oplus \mathcal{O}_C(B)]\}$;*

(b) *If $L = K_C$, as a degree 2 regular plane curve,*

$$W^1_{C,2}(K_C) = W^0_{C,2}(L)^{SS} = \{[\mathcal{O}_C(P) \oplus \mathcal{O}_C(\iota(P))] : P \in C\} \simeq \mathbb{P}^1.$$

Next, we study the infinitesimal structures of non-abelian Brill-Noether loci. Set

$$W^0_{C,2}(L)^s := W^0_{C,2}(L) \backslash W^0_{C,2}(L)^{\mathrm{SS}}.$$

Then the infinitesimal structure of $W^0_{C,2}(L)$ at points $[V] \in W^0_{C,2}(L)^s$ is simple: each $[V]$ consists a single stable rank two vector bundle with $\det(V) = L, h^0(C, V) = 1$ and $\mathrm{Aut}(V) \simeq \mathbb{F}_q^*$.

Now we consider $W^0_{C,2}(L)^{\mathrm{SS}}$: (a) $L \neq K_C$. Then there exist two points A, B of C such that $L = \mathcal{O}_C(A+B)$. Thus, for any $V \in [\mathcal{O}_C(P) \oplus \mathcal{O}_C(A+B-P)] \notin W^1_{C,2}(L)$, V is given by an extension $0 \to \mathcal{O}_C(P) \to V \to \mathcal{O}_C(A+B-P) \to 0$ due to the fact that for the nontrivial extension $0 \to \mathcal{O}_C(A+B-P) \to W \to \mathcal{O}_C(P) \to 0$, $h^0(C,W) = 0$. Thus, each class $[\mathcal{O}_C(P) \oplus \mathcal{O}_C(A+B-P)] \notin W^1_{C,2}(L)$ consists of exactly two vector bundles, i.e., $V_1 = \mathcal{O}_C(P) \oplus \mathcal{O}_C(A+B-P)$ and V_2 given by the nontrivial extension $0 \to \mathcal{O}_C(P) \to V \to \mathcal{O}_C(A+B-P) \to 0$. Clearly, $h^0(C,V_1) = h^0(C,V_2) = 1$ and $\#\,\mathrm{Aut}(V_1) = (q-1)^2, \#\mathrm{Aut}(V_2) = q-1$.

To study $W^1_{C,2}(L) = \{[\mathcal{O}_C(A) \oplus \mathcal{O}_C(B)]\}$, we divide it into two subcases: (i) $A \neq B$. Then there are exactly three vector bundles V_0, V_1 and V_2 in the class $[\mathcal{O}_C(A) \oplus \mathcal{O}_C(B)]$. They are $V_0 = \mathcal{O}_C(A) \oplus \mathcal{O}_C(B)$, V_1 given by the nontrivial extension $0 \to \mathcal{O}_C(A) \to V_1 \to \mathcal{O}_C(B) \to 0$ and V_2 given by the nontrivial extension $0 \to \mathcal{O}_C(B) \to V_2 \to \mathcal{O}_C(A) \to 0$. Clearly, $h^0(C,V_1) = 2, h^0(C,V_1) = h^0(C,V_2) = 1$ and $\#\mathrm{Aut}(V_0) = (q-1)^2,\ \#\mathrm{Aut}(V_1) = \#\mathrm{Aut}(V_2) = q-1$.

Thus in particular,

$$\begin{aligned}\gamma_{C,2,\mathbb{F}_q}(L) \;=\; &(q^2+q+1-(N_1-1))\cdot\frac{q-1}{q-1}\\ &+(N_1-2)\left(\frac{q-1}{(q-1)^2}+\frac{q-1}{q-1}\right)+\left(\frac{q^2-1}{(q-1)^2}+\frac{q-1}{q-1}+\frac{q-1}{q-1}\right).\end{aligned}$$

Here $N_1 = q+1-(\omega_1+\ldots+\omega_4)$ denotes the number of $\mathbb{F}_q$-rational points of C.

(ii) $A = B$. Then the infinitesimal structure at $[\mathcal{O}_C(A) \oplus \mathcal{O}_C(A)]$ is as follows: an independent point corresponding to $V_0 = \mathcal{O}_C(A) \oplus \mathcal{O}_C(A)$ and a projective line parametrizing all nontrivial extension $0 \to \mathcal{O}_C(A) \to V \to \mathcal{O}_C(A) \to 0$. Clearly, $h^0(C,V_0) = 2, h^0(C,V) = 1$ and $\#\mathrm{Aut}(V_0) = (q^2-1)(q^2-q),\ \#\mathrm{Aut}(V) = q(q-1)$;

Thus in particular,

$$\begin{aligned}\gamma_{C,2,\mathbb{F}_q}(L) \;=\; &(q^2+q+1-(N_1-1))\cdot\frac{q-1}{q-1}\\ &+(N_1-2)\left(\frac{q-1}{(q-1)^2}+\frac{q-1}{q-1}\right)+\left(\frac{q^2-1}{(q^2-1)(q^2-q)}+(q+1)\frac{q-1}{q(q-1)}\right).\end{aligned}$$

(b) $L = K_C$. Then $K_C = \mathcal{O}_C(P+\iota(P))$ for all points P. Therefore for all $[V] \in W^1_{C,2}(L) = W^0_{C,2}(L)^{\mathrm{ss}}$, $[V] = [\mathcal{O}_C(P) \oplus \mathcal{O}_C(\iota(P))]$. Accordingly, two subcases: (i) $P \neq \iota P$. Then there are exactly three vector bundles V_0, V_1 and V_2 in the class $[\mathcal{O}_C(P) \oplus \mathcal{O}_C(\iota(P)]$. They are $V_0 = \mathcal{O}_C(P) \oplus \mathcal{O}_C(\iota(P))$, V_1 given by the nontrivial extension $0 \to \mathcal{O}_C(P) \to V_1 \to \mathcal{O}_C(\iota(P)) \to 0$ and V_2 given by the nontrivial extension $0 \to \mathcal{O}_C(\iota(P)) \to V_2 \to \mathcal{O}_C(P) \to 0$. Clearly, $h^0(C,V_0) = 2, h^0(C,V_1) = h^0(C,V_2) = 1$ and $\#\mathrm{Aut}(V_0) = (q-1)^2,\ \#\mathrm{Aut}(V_1) = \#\mathrm{Aut}(V_2) = q-1$;

(ii) $P = \iota(P)$ a Weierstrass point, all of which are six. Then the infinitesimal structure at $[\mathcal{O}_C(P) \oplus \mathcal{O}_C(P)]$ is as follows: an independent point corresponding to $V_0 = \mathcal{O}_C(P) \oplus \mathcal{O}_C(P)$ and a projective line parametrizing all nontrivial extension $0 \to \mathcal{O}_C(P) \to V \to \mathcal{O}_C(P) \to 0$. Clearly, $h^0(C,V_0) = 2, h^0(C,V) = 1$ and $\#\mathrm{Aut}(V_0) = (q^2-1)(q^2-q),\ \#\mathrm{Aut}(V) = q(q-1)$.

Thus, in particular,

$$\gamma_{C,2,\mathbb{F}_q}(K_C) = (q^2+q+1-(q+1))\cdot\frac{q-1}{q-1}+(q+1-6)\left(\frac{q^2-1}{(q-1)^2}+\frac{q-1}{q-1}+\frac{q-1}{q-1}\right)$$
$$+\,6\left(\frac{q^2-1}{(q^2-1)(q^2-q)}+(q+1)\frac{q-1}{q(q-1)}\right).$$

All in all, we have completed the proof of the following:

PROPOSITION. *With the same notation as above:*
(a) *For* $L\neq K_C$, (i) *if* $L\notin\Delta$, $\gamma_{C,2,\mathbb{F}_q}(L)=\frac{q^3+2q-3+N_1}{q-1}$; (ii) *if* $L\in\Delta$, $\gamma_{C,2,\mathbb{F}_q}(L)=\frac{q^3-2+N_1}{q-1}$;
(b) *if* $L=K_C$, $\gamma_{C,2,\mathbb{F}_q}(L)=\frac{q^3+2q^2-10q+5}{q-1}$.
In particular,

$$\gamma_{C,2,\mathbb{F}_q}(2) = \left(\prod_{i=1}^{4}(1-\omega_i)-(q+1)\right)\cdot\frac{q^3+2q-3+N_1}{q-1}$$
$$+\,q\cdot\frac{q^3-2+N_1}{q-1}+\frac{q^3+2q^2-10q+5}{q-1}.$$

In this way, by using the ugly formula in 2.1.3, we can finally write down the rank two non-abelian zeta functions for genus two curves, where degree 8 polynomials are involved. We leave this to the reader.

Chapter II. Non-abelian L-functions. While we may introduce general non-abelian L-functions by using more general test functions as Tate did in his Thesis where an abelian version is discussed, in this paper, we decide to take a different approach using Eisenstein series. (We remind the reader that for the abelian picture, Eisenstein series are not available.) Moreover, as for the integration domain, we use a much more general type of moduli spaces.

II.1. Epstein zeta functions and non-abelian zeta functions. To motivate what follows, we begin this chapter with a discussion on non-ablian zeta functions for number fields.

For simplicity, assume that the number field involved is the field of rationals. A lattice Λ over $\mathbb{Q}$ is semi-stable, by definition, if for any sublattice Λ_1 of Λ,

$$(\mathrm{Vol}\,\Lambda_1)^{\mathrm{rank}\,\Lambda}\geq(\mathrm{Vol}\,\Lambda)^{\mathrm{rank}\,\Lambda_1}.$$

Denote the moduli space of rank r semistable lattices over $\mathbb{Q}$ by $\mathcal{M}_{\mathbb{Q},r}$, then the lattice version of *rank r non-abelian zeta function* $\xi_{\mathbb{Q},r}(s)$ of $\mathbb{Q}$ is defined to be

$$\xi_{\mathbb{Q},r}(s) := \int_{\mathcal{M}_{\mathbb{Q},r}} \left(e^{h^0(\mathbb{Q},\Lambda)} - 1\right) \cdot \left(e^{-s}\right)^{\deg(\Lambda)} d\mu(\Lambda), \qquad \mathrm{Re}(s) > 1,$$

where $h^0(\mathbb{Q},\Lambda) := \log\left(\sum_{x\in\Lambda} \exp\left(-\pi|x|^2\right)\right)$ and $\deg(\Lambda) := -\log\Big(\mathrm{Vol}(\mathbb{R}^{\mathrm{rank}(\Lambda)}/\Lambda)\Big)$ denotes the Arakelov degree of Λ. Moreover, note that the newly defined h^0 has a natural company h^1 and that similarly as cohomology for bundles over curves, h^i satisfy the Serre duality and Riemann-Roch (for details, see [We2]). In particular, as shown in [We2], (see also the calculation below for an alternative proof):

(i) $\xi_{\mathbb{Q},1}(s)$ coincides with the (completed) Riemann-zeta function;

(ii) $\xi_{\mathbb{Q},r}(s)$ can be meromorphically extended to the whole complex plane;

(iii) $\xi_{\mathbb{Q},r}(s)$ satisfies the functional equation

$$\xi_{\mathbb{Q},r}(s) = \xi_{\mathbb{Q},r}(1-s);$$

(iv) $\xi_{\mathbb{Q},r}(s)$ has only two singularities, simple poles, at $s = 0, 1$, with the same residues $\mathrm{Vol}\left(\mathcal{M}_{\mathbb{Q},r}[1]\right)$, the Tamagawa type volume of the space of rank r semi-stable lattice of volume 1.

Denote by $\mathcal{M}_{\mathbb{Q},r}[T]$ the moduli space of rank r semi-stable lattices of volume T. We have a trivial decomposition

$$\mathcal{M}_{\mathbb{Q},r} = \cup_{T>0}\mathcal{M}_{\mathbb{Q},r}[T].$$

Moreover, there is a natural morphism

$$\mathcal{M}_{\mathbb{Q},r}[T] \to \mathcal{M}_{\mathbb{Q},r}[1], \qquad \Lambda \mapsto T^{\frac{1}{r}} \cdot \Lambda.$$

With this, for $\mathrm{Re}(s) > 1$,

$$\begin{aligned}\xi_{\mathbb{Q},r}(s) &= \int_{\cup_{T>0}\mathcal{M}_{\mathbb{Q},r}[T]} \left(e^{h^0(\mathbb{Q},\Lambda)} - 1\right) \cdot \left(e^{-s}\right)^{\deg(\Lambda)} d\mu(\Lambda) \\ &= \int_0^\infty T^s \frac{dT}{T} \int_{\mathcal{M}_{\mathbb{Q},r}[1]} \left(e^{h^0(\mathbb{Q},T^{\frac{1}{r}}\cdot\Lambda)} - 1\right) \cdot d\mu_1(\Lambda),\end{aligned}$$

where $d\mu_1$ denotes the induced Tamagawa measure on $\mathcal{M}_{\mathbb{Q},r}[1]$.

Thus note that

$$h^0(\mathbb{Q}, T^{\frac{1}{r}} \cdot \Lambda) = \log\left(\sum_{x\in\Lambda} \exp\left(-\pi|x|^2 \cdot T^{\frac{2}{r}}\right)\right),$$

and for $B \neq 0$,

$$\int_0^\infty e^{-AT^B} T^s \frac{dT}{T} = \frac{1}{B} \cdot A^{-\frac{s}{B}} \cdot \Gamma(\frac{s}{B}),$$

we have

$$\xi_{\mathbb{Q},r}(s) = \frac{r}{2} \cdot \pi^{-\frac{r}{2}s} \Gamma(\frac{r}{2}s) \cdot \int_{\mathcal{M}_{\mathbb{Q},r}[1]} \left(\sum_{x \in \Lambda \backslash \{0\}} |x|^{-rs} \right) \cdot d\mu_1(\Lambda).$$

Set now the completed Epstein zeta function, a special kind of Eisenstein series, associated to the rank r lattice Λ over $\mathbb{Q}$ by

$$\hat{E}(\Lambda; s) := \pi^{-s} \Gamma(s) \cdot \sum_{x \in \Lambda \backslash \{0\}} |x|^{-2s},$$

then we have the following

PROPOSITION. (Eisenstein series and Non-Abelian Zeta Functions) *With the same notation as above,*

$$\xi_{\mathbb{Q},r}(s) = \frac{r}{2} \int_{\mathcal{M}_{\mathbb{Q},r}[1]} \hat{E}\left(\Lambda, \frac{r}{2}s\right) d\mu_1(\Lambda).$$

Remark. Such a non-abelian zeta is indeed very beautiful: not only its construction is so elegent, its structure is also very rational. Recently, Lagarias and Suzuki [LS] have shown that the rank two zeta $\xi_{\mathbb{Q},2}(s)$ for the field of rationals satisfies the Riemann Hypothesis, i.e., the zeros are all on the line $\Re(s) = \frac{1}{2}$.

II.2. Canonical polygons and geometric truncation. We start with Weil's adelic interpretation of locally free sheaves on curves. Fix a smooth geometrically connected projective curve X over a finite field $\mathbb{F}_q$. Denote its function field by F and identify the places of F with the closed points of X which we denote by $|X|$. For each place x of F, set F_x the x-completion of F with $\mathcal{O}_x$ the ring of integers, π_x a local parameter, and $\kappa(x)$ the residue field. Denote by $x : F_x^* \to \mathbb{Z}$ the normalized valuation of F_x such that $x(\pi_x) = 1$. Denote also by $\mathbb{A}$ the ring of adeles and $\mathcal{O}_{\mathbb{A}}$ the ring of integers.

If E is a locally free $\mathcal{O}_F$-sheaf of rank r over X, denote by E_F the fiber of E at the generic point $\mathrm{Spec}(F)$ of X (E_F is an F-vector space of dimension r), and for each $v \in |X|$, set $E_{\mathcal{O}_v} := H^0(\mathrm{Spec}\, \mathcal{O}_{F_v}, E)$ a free $\mathcal{O}_v$-module of rank r. In particular, we have a canonical isomorphism:

$$\mathrm{can}_v : F_v \otimes_{\mathcal{O}_v} E_{\mathcal{O}_v} \simeq F_v \otimes_F E_F.$$

Thus, in particular, with respect to a basis $\alpha_F : F^r \simeq E_F$ of its generic fiber and a basis $\alpha_{\mathcal{O}_v} : \mathcal{O}_v^r \simeq E_{\mathcal{O}_v}$ for any $v \in |X|$, the elements $g_v := (F_v \otimes_F \alpha_F)^{-1} \circ \mathrm{can}_v \circ (F_v \otimes_{\mathcal{O}_v} \alpha_{\mathcal{O}_v}) \in GL_r(F_v)$ for all $v \in |X|$ define an element $g_{\mathbb{A}} := (g_v)_{v\in|X|}$ of $GL_r(\mathbb{A})$, since for almost all v we have $g_v \in GL_r(\mathcal{O}_v)$. As a result, we obtain a bijection from the set of isomorphism classes of triples $(E; \alpha_F; (\alpha_{\mathcal{O}_v})_{v\in|X|})$ as above onto $GL_r(\mathbb{A})$. Moreover, if $r \in GL_r(F), k \in GL_r(\mathcal{O}_F)$ and if this bijection maps the triple $(E; \alpha_F; (\alpha_{\mathcal{O}_v})_{v\in|X|})$ onto $g_{\mathbb{A}}$, the same map maps the triple $(E; \alpha_F \circ r^{-1}; (\alpha_{\mathcal{O}_v} \circ k_v)_{v\in|X|})$ onto $rg_{\mathbb{A}}k$. Therefore the above bijection induces a bijection between the set of isomorphism classes of locally free $\mathcal{O}_F$-sheaves of rank r over X and the double coset space $GL_r(F)\backslash GL_r(\mathbb{A})/GL_r(\mathcal{O}_F)$.

More generally, let $r = r_1+\cdots+r_s$ be a partition $I = (r_1, \ldots, r_s)$ of r and let P_I be the corresponding standard parabolic subgroup of GL_r. Then we have a natural bijection from the set of isomorphism classes of triple $(E_*; \alpha_{*,F} : (\alpha_{*,\mathcal{O}_v})_{v\in|X|})$ onto $P_I(\mathbb{A})$, where $E_* := ((0) = E_0 \subset E_1 \subset \cdots \subset E_s)$ is a filtration of locally free sheaves of rank $(r_1, r_1 + r_2, \ldots, r_1 + r_2 + \cdots + r_s = r)$ over X (i.e, each E_j is a vector sheaf of rank $r_1 + r_2 + \cdots + r_j$ over X and each quotient E_j/E_{j-1} is torsion free), which is equipped with an isomorphism of filtrations of F-vector spaces

$$\alpha_{*,F} : \big((0) = F_0 \subset F^{r_1} \subset \cdots \subset F^{r_1+r_2+\cdots+r_s=r}\big) \simeq (E_*)_F,$$

and with an isomorphism of filtrations of free $\mathcal{O}_v$-modules

$$\alpha_{*,\mathcal{O}_v} : \big((0) \subset \mathcal{O}_v^{r_1} \subset \cdots \subset \mathcal{O}_v^{r_1+r_2+\cdots+r_s=r}\big) \simeq (E_*)_{\mathcal{O}_v},$$

for every $v \in |X|$. Moreover this bijection induces a bijection between the set of isomorphism classes of the filtrations of locally free sheaves of rank $(r_1, r_1 + r_2, \cdots, r_1+r_2+\cdots+r_s = r)$ over X and the double coset space $P_I(F)\backslash P_I(\mathbb{A})/P_I(\mathcal{O}_{\mathbb{A}})$. The natural embedding $P_I(\mathbb{A}) \hookrightarrow P_I(\mathbb{A})$ (resp. the canonical projection $P_I(\mathbb{A}) \to M_I(\mathbb{A}) \to GL_{r_j}(\mathbb{A})$ for $j = 1, \ldots, s$, where M_I denotes the standard Levi of P_I) admits the modular interpretation

$$(E_*; \alpha_{*,F} : (\alpha_{*,\mathcal{O}_v})_{v\in|X|}) \mapsto (E_s; \alpha_{s,F} : (\alpha_{s,\mathcal{O}_v})_{v\in|X|})$$

(resp.

$$(E_*; \alpha_{*,F} : (\alpha_{*,\mathcal{O}_v})_{v\in|X|}) \mapsto (\mathrm{gr}_j(E_*); \mathrm{gr}_j(\alpha_{*,F}), \mathrm{gr}_j(\alpha_{*,\mathcal{O}_v})_{v\in|X|}),$$

where $\mathrm{gr}_j(E_*) := E_j/E_{j-1}$, $\mathrm{gr}_j(\alpha_{*,F}) : F^{r_j} \simeq \mathrm{gr}_j(E_*)_F$ and $\mathrm{gr}_j(\alpha_{*,\mathcal{O}_v}) : {\mathcal{O}_v}^{r_j} \simeq \mathrm{gr}_j(E_*)_{\mathcal{O}_v}$, $v \in |X|$ are induced by $\alpha_{*,F}$ and $\alpha_{*,\mathcal{O}_v}$ respectively.)

Denote by E_g the rank r locally free sheaf on X associated to $g \in GL_r(\mathbb{A})$. Then,

$$\deg(E_g) = -\log\big(N(\det g)\big)$$

with $N : GL_1(\mathbb{A}_F) = \mathbb{I}_F \to \mathbb{Q}_{>0}$ the standard norm map of the idelic group of F.

With this, for $g \in GL_r(\mathbb{A})$ and a parabolic subgroup Q of GL_r, denote by $E_*^{g;Q}$ the filtration of the locally free sheaf E_g induced by the parabolic subgroup Q.

Now following Lafforgue [Laf], introduce an associated polygon $p_Q^g : [0, r] \to \mathbb{Q}$ by the following 3 conditions:

(i) $p_Q^g(0) = p_Q^g(r) = 0$;

(ii) p_Q^g is affine on the interval $[\mathrm{rank} E_{i-1}^{g;Q}, \mathrm{rank} E_i^{g;Q}]$; and

(iii) for all indices i,

$$p_Q^g(\mathrm{rank} E_i^{g;Q}) = \deg(E_i^{g;Q}, \rho_i^{g;Q}) - \frac{\mathrm{rank} E_i^{g;Q}}{r} \cdot \deg(E_g, \rho_g).$$

Then by Prop. 1 in I.1.1, i.e., the existence and uniqueness of Harder-Narasimhan filtration, there is a unique convex polygon $\bar{p}^g$ which bounds all p_Q^g from above for all parabolic subgroups Q for GL_r. Moreover there exists a parabolic subgroup $\bar{Q}^g$ such that $p_{\bar{Q}^g}^g = \bar{p}^g$. In particular, as a direct consequence, we obtain the following well-known:

LEMMA. (See e.g. [Laf].) *For any fixed polygon $p : [0, r] \to \mathbb{Q}$ and any $d \in \mathbb{Z}$, the subset*

$$\{g \in GL_r(F)\backslash GL_r(\mathbb{A}) : deg\, g = d, \bar{p}^g \leq p\}$$

is compact.

Similarly yet more generally, for a fixed parabolic subgroup P of GL_r and $g \in GL_r(\mathbb{A})$, there is a unique maximal element $\bar{p}_P^g$ among all p_Q^g, where Q runs over all parabolic subgroups of GL_r which are contained in P. And we have:

LEMMA$'$. (See e.g. [Laf].) *For any fixed polygon $p : [0, r] \to \mathbb{Q}$, $d \in \mathbb{Z}$ and any standard parabolic subgroup P of GL_r, the subset*

$$\{g \in GL_r(F)\backslash GL_r(\mathbb{A}) : deg\, g = d, \bar{p}_P^g \leq p, p_P^g \geq -p\}$$

is compact.

Moreover, let $p, q : [0, r] \to \mathbb{R}$ be two polygons and P a standard parabolic subgroup of GL_r. Then as in [Laf], we say $q >_P p$ if for any $1 \leq i \leq |P|$,

$$q(\mathrm{rank} E_i^P) > p(\mathrm{rank} E_i^P)$$

where $(r_1, \dots, r_{|P|})$ denotes the partition of r corresponding to P. As usual denote by 1 the characteristic function of the variable $g \in GL_r(\mathbb{A})$. For example,

$$1(\bar{p}^g \leq p)(g) = \begin{cases} 1, & \text{if } p^g \leq p \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the following result of Lafforgue:

PROPOSITION. [Laf, Prop. V.1.c] *For any convex polygon $p : [0, r] \to \mathbb{R}$, as a function of $g \in GL_r(\mathbb{A})$,*

$$1(\bar{p}^g \leq p) = \sum_{P \supset P_0} (-1)^{|P|-1} \sum_{\delta \in P(F) \backslash GL_r(F)} 1(p_P^{\delta g} >_P p).$$

Here P runs over all standard parabolic subgpoups of GL_r.

II.3. Non-abelian L-functions. In this section, we introduce non-abelian L-functions for function fields and study their basic properties.

3.1. Choice of moduli spaces. For the function field F with genus g_X, and for a fixed $r \in \mathbb{Z}_{>0}$, we take the moduli space to be

$$\mathcal{M}_{F,r}^{\leq p} := \{g \in GL_r(F) Z_{GL_r(\mathbb{A})} \backslash GL_r(\mathbb{A}) : \bar{p}^g \leq p\}$$

for a fixed convex polygon $p : [0, r] \to \mathbb{R}$. Also we denote by $d\mu$ the induced Tamagawa measures on $\mathcal{M}_{F,r}^{\leq p}$.

More generally, for any standard parabolic subgroup P of GL_r, we introduce the moduli spaces

$$\mathcal{M}_{F,r}^{P;\leq p} := \{g \in P(F) Z_{GL_r(\mathbb{A})} \backslash GL_r(\mathbb{A}) : \bar{p}_P^g \leq p, \bar{p}_P^g \geq -p\}.$$

By the discussion in II.2, these moduli spaces $\mathcal{M}_{F,r}^{P;\leq p}$ are all compact, a key property which plays a central role in our definition of non-abelian L-functions below.

3.2. Choice of Eisenstein series: first approach to non-abelian *L*-function. To facilitate our ensuing discussion, we start with some preparations. For details, please consult [MW], which is heavily used in this subsection. (The experienced reader may skip this subsection, except for possible later reference about notations.)

Fix a connected reduction group G defined over F, denote by Z_G its center. Fix a minimal parabolic subgroup P_0 of G. Then $P_0 = M_0 U_0$, where as usual we fix once and for all the Levi M_0 and the unipotent radical U_0. A parabolic subgroup P of G is called standard if $P \supset P_0$. For such groups write $P = MU$

with $M_0 \subset M$ the standard Levi and U the unipotent radical. Denote by $\mathrm{Rat}(M)$ the group of rational characters of M, i.e, the morphism $M \to \mathbb{G}_m$ where $\mathbb{G}_m$ denotes the multiplicative group. Set

$$\mathfrak{a}_M^* := \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{C}, \qquad \mathfrak{a}_M := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(M), \mathbb{C}),$$

and

$$\mathrm{Re}\, \mathfrak{a}_M^* := \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{R}, \qquad \mathrm{Re}\, \mathfrak{a}_M := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(M), \mathbb{R}).$$

For any $\chi \in \mathrm{Rat}(M)$, we obtain a (real) character $|\chi| : M(\mathbb{A}) \to \mathbb{R}^*$ defined by $m = (m_v) \mapsto m^{|\chi|} := \prod_{v \in S} |m_v|_v^{\chi_v}$ with $|\cdot|_v$ the v-absolute values. Set then $M(\mathbb{A})^1 := \cap_{\chi \in \mathrm{Rat}(M)} \mathrm{Ker}|\chi|$, which is a normal subgroup of $M(\mathbb{A})$. Set X_M to be the group of complex characters which are trivial on $M(\mathbb{A})^1$. Denote by $H_M := \log_M : M(\mathbb{A}) \to \mathfrak{a}_M$ the map such that $\forall \chi \in \mathrm{Rat}(M) \subset \mathfrak{a}_M^*, \langle \chi, \log_M(m) \rangle := \log(m^{|\chi|})$. Clearly,

$$M(\mathbb{A})^1 = \mathrm{Ker}(\log_M); \qquad \log_M(M(\mathbb{A})/M(\mathbb{A})^1) \simeq \mathrm{Re}\, \mathfrak{a}_M.$$

Hence in particular there is a natural isomorphism $\kappa : \mathfrak{a}_M^* \simeq X_M$. Set

$$\mathrm{Re}\, X_M := \kappa(\mathrm{Re}\, \mathfrak{a}_M^*), \qquad \mathrm{Im}\, X_M := \kappa(i \cdot \mathrm{Re}\, \mathfrak{a}_M^*).$$

Moreover define our working space X_M^G to be the subgroup of X_M consisting of complex characters of $M(\mathbb{A})/M(\mathbb{A})^1$ which are trivial on $Z_{G(\mathbb{A})}$.

Fix a maximal compact subgroup $\mathbb{K}$ such that for all standard parabolic subgroups $P = MU$ as above, $P(\mathbb{A}) \cap \mathbb{K} = M(\mathbb{A}) \cap \mathbb{K} \cdot U(\mathbb{A}) \cap \mathbb{K}$. Hence we get the Langlands decomposition $G(\mathbb{A}) = M(\mathbb{A}) \cdot U(\mathbb{A}) \cdot \mathbb{K}$. Denote by $m_P : G(\mathbb{A}) \to M(\mathbb{A})/M(\mathbb{A})^1$ the map $g = m \cdot n \cdot k \mapsto M(\mathbb{A})^1 \cdot m$ where $g \in G(\mathbb{A}), m \in M(\mathbb{A}), n \in U(\mathbb{A})$ and $k \in \mathbb{K}$.

Fix Haar measures on $M_0(\mathbb{A}), U_0(\mathbb{A}), \mathbb{K}$ respectively such that:

(1) the induced measure on $M(F)$ is the counting measure and the volume of the induced measure on $M(F)\backslash M(\mathbb{A})^1$ is 1. (Recall that it is a fundamental fact that $M(F)\backslash M(\mathbb{A})^1$ is compact.)

(2) the induced measure on $U_0(F)$ is the counting measure and the volume of $U(F)\backslash U_0(\mathbb{A})$ is 1. (Recall that being unipotent radical, $U(F)\backslash U_0(\mathbb{A})$ is compact.)

(3) the volume of $\mathbb{K}$ is 1.

Such measures then also induce Haar measures via $\log_M$ to $\mathfrak{a}_{M_0}, \mathfrak{a}_{M_0}^*$, etc. Furthermore, if we denote by ρ_0 a half of the sum of the positive roots of the maximal split torus T_0 of the central Z_{M_0} of M_0, then

$$f \mapsto \int_{M_0(\mathbb{A}) \cdot U_0(\mathbb{A}) \cdot \mathbb{K}} f(mnk)\, dk\, dn\, m^{-2\rho_0}\, dm$$

defined for continuous functions with compact supports on $G(\mathbb{A})$ defines a Haar measure dg on $G(\mathbb{A})$. This in turn gives measures on $M(\mathbb{A}), U(\mathbb{A})$ and hence on $\mathfrak{a}_M, \mathfrak{a}_M^*$, $P(\mathbb{A})$, etc, for all parabolic subgroups P. In particular, one checks that the following compactibility condition holds

$$\int_{M_0(\mathbb{A})\cdot U_0(\mathbb{A})\cdot\mathbb{K}} f(mnk)\, dk\, dn\, m^{-2\rho_0} dm = \int_{M(\mathbb{A})\cdot U(\mathbb{A})\cdot\mathbb{K}} f(mnk)\, dk\, dn\, m^{-2\rho_P} dm$$

for all continuous functions f with compact supports on $G(\mathbb{A})$, where ρ_P denotes a half of the sum of the positive roots of the maximal split torus T_P of the central Z_M of M. For later use, denote also by Δ_P the set of positive roots determined by (P, T_P), $\Delta_0 = \Delta_{P_0}$ and W the associated Weyl group.

Fix an isomorphism $T_0 \simeq \mathbb{G}_m^R$ and a place υ_0 of F and a uniformizer π_{υ_0} at υ_0. The group $\pi_{\upsilon_0}^{\mathbb{Z}}$ generated by π_{υ_0} can be identified with a subgroup of $\mathbb{A}^*$ and hence $(\pi_{\upsilon_0}^{\mathbb{Z}})^R$ with a subgroup of $T_0(\mathbb{A})$. Thus there exists a W-invariant subgroup of $Z_{M_0(\mathbb{A})}$ which is isomorphic to a subgroup of finite index of $(\pi_{\upsilon_0}^{\mathbb{Z}})^R$. Fix such a group once and for all and denote it by $A_{M_0(\mathbb{A})}$.

More generally, for a standard parabolic subgroup $P = MU$, set $A_{M(\mathbb{A})} := A_{M_0(\mathbb{A})} \cap Z_{M(\mathbb{A})}$ where as used above Z_* denotes the center of the group $*$. Then $A_{M(\mathbb{A})}\backslash M(\mathbb{A})/M(\mathbb{A})^1$ is finite. For later use, set also $A_{M(\mathbb{A})}^G := \{a \in A_{M(\mathbb{A})} : \log_G a = 0\}$. Then $A_{M(\mathbb{A})}$ contains $A_{G(\mathbb{A})} \oplus A_{M(\mathbb{A})}^G$ as a subgroup of finite index.

Note that $\mathbb{K}$, $M(F)\backslash M(\mathbb{A})^1$ and $U(F)\backslash U(\mathbb{A})$ are all compact, thus with the Langlands decomposition $G(\mathbb{A}) = U(\mathbb{A})M(\mathbb{A})\mathbb{K}$ in mind, the reduction theory for $G(F)\backslash G(\mathbb{A})$ or more generally $P(F)\backslash G(\mathbb{A})$ is reduced to that for $A_{M(\mathbb{A})}$. As such for $t_0 \in M_0(\mathbb{A})$, set

$$A_{M_0(\mathbb{A})}(t_0) := \{a \in A_{M_0(\mathbb{A})} : a^\alpha > t_0^\alpha\ \forall \alpha \in \Delta_0\}.$$

Then, for a fixed compact subset $\omega \subset P_0(\mathbb{A})$, we have the corresponding Siegel set

$$S(\omega; t_0) := \{p \cdot a \cdot k : p \in \omega, a \in A_{M_0(\mathbb{A})}(t_0), k \in \mathbb{K}\}.$$

In particular, for big enough ω and small enough t_0, i.e, t_0^α is very close to 0 for all $\alpha \in \Delta_0$, the classical reduction theory may be restated as $G(\mathbb{A}) = G(F) \cdot S(\omega; t_0)$. More generally set

$$A_{M_0(\mathbb{A})}^P(t_0) := \{a \in A_{M_0(\mathbb{A})} : a^\alpha > t_0^\alpha,\ \forall \alpha \in \Delta_0^P\},$$

and

$$S^P(\omega; t_0) := \{p \cdot a \cdot k : p \in \omega, a \in A_{M_0(\mathbb{A})}^P(t_0), k \in \mathbb{K}\}.$$

Then similarly as above for big enough ω and small enough t_0, $G(\mathbb{A}) = P(F) \cdot S^P(\omega; t_0)$. (Here Δ_0^P denotes the set of positive roots for $(P_0 \cap M, T_0)$.)

Fix an embedding $i_G : G \hookrightarrow SL_n$ sending g to (g_{ij}). Introducing a hight function on $G(\mathbb{A})$ by setting $\|g\| := \prod_{v \in S} \sup\{|g_{ij}|_v : \forall i, j\}$. It is well known that up to $O(1)$, hight functions are unique. This implies that the following growth conditions do not depend on the height function we choose.

A function $f : G(\mathbb{A}) \to \mathbb{C}$ is said to have moderate growth if there exist $c, r \in \mathbb{R}$ such that $|f(g)| \leq c \cdot \|g\|^r$ for all $g \in G(\mathbb{A})$. Similarly, for a standard parabolic subgroup $P = MU$, a function $f : U(\mathbb{A})M(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ is said to have moderate growth if there exist $c, r \in \mathbb{R}, \lambda \in \mathrm{Re}X_{M_0}$ such that for any $a \in A_{M(\mathbb{A})}, k \in \mathbb{K}, m \in M(\mathbb{A})^1 \cap S^P(\omega; t_0)$,

$$|f(amk)| \leq c \cdot \|a\|^r \cdot m_{P_0}(m)^\lambda.$$

Now fix a place v_0 of F, denote by $G(\mathbb{A})_{v_0}$ the inverse image of $G(F_{v_0})$ in $G(\mathbb{A})$. Denote by $\mathfrak{z}$ the Bernstein centre of $G(\mathbb{A})_{v_0}$. The $\mathfrak{z}$ acts naturally on the locally constant functions on $G(\mathbb{A})$.

By definition, a function $\phi : U(\mathbb{A})M(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ is called automorphic if:

(i) ϕ has moderate growth;

(ii) ϕ is locally constant;

(iii) ϕ is $\mathbb{K}$-finite, i.e., the $\mathbb{C}$-span of all $\phi(k_1 \cdot * \cdot k_2)$ parametrized by $(k_1, k_2) \in \mathbb{K} \times \mathbb{K}$ is finite dimensional;

(iv) ϕ is $\mathfrak{z}$-finite, i.e., the $\mathbb{C}$-span of all $\delta(X)\phi$ parametrized by all $X \in \mathfrak{z}$ is finite dimensional.

For such a function ϕ, set $\phi_k : M(F)\backslash M(\mathbb{A}) \to \mathbb{C}$ to be the function defined by $m \mapsto m^{-\rho_P}\phi(mk)$ for all $k \in \mathbb{K}$. Set $A(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ be the space of automorphic forms on $U(\mathbb{A})M(F)\backslash G(\mathbb{A})$.

For a measurable locally L^1-function $f : U(F)\backslash G(\mathbb{A}) \to \mathbb{C}$, define its constant term along with the standard parabolic subgroup $P = UM$ to be the function $f_P : U(\mathbb{A})\backslash G(\mathbb{A}) \to \mathbb{C}$ given by $g \to \int_{U(F)\backslash G(\mathbb{A})} f(ng)dn$. Then an automorphic form $\phi \in A(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ is called a cusp form if for any standard parabolic subgroup P' properly contained in P, $\phi_{P'} \equiv 0$. Denote by $A_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ the space of cusp forms on $U(\mathbb{A})M(F)\backslash G(\mathbb{A})$. One checks easily that:

(i) all cusp forms are rapidly decreasing; and hence

(ii) there is a natural pairing

$$\langle \cdot, \cdot \rangle : A_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A})) \times A(U(\mathbb{A})M(F)\backslash G(\mathbb{A})) \to \mathbb{C}$$

defined by $\langle \psi, \phi \rangle := \int_{Z_{M(\mathbb{A})}U(\mathbb{A})M(F)\backslash G(\mathbb{A})} \psi(g)\bar{\phi}(g)\, dg$.

Moreover, for a (complex) character $\xi : Z_{M(\mathbb{A})} \to \mathbb{C}^*$ of $Z_{M(\mathbb{A})}$ set

$$\begin{aligned} &A(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\xi \\ &\quad := \{\phi \in A(U(\mathbb{A})M(F)\backslash G(\mathbb{A})) : \phi(zg) = z^{\rho_P} \cdot \xi(z) \cdot \phi(g), \forall z \in Z_{M(\mathbb{A})}, g \in G(\mathbb{A})\} \end{aligned}$$

and

$$A_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\xi := A_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A})) \cap A(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\xi.$$

Now set

$$A(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_Z := \sum_{\xi \in \mathrm{Hom}(Z_{M(\mathbb{A})},\mathbb{C}^*)} A(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\xi$$

and

$$A_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_Z := \sum_{\xi \in \mathrm{Hom}(Z_{M(\mathbb{A})},\mathbb{C}^*)} A_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\xi.$$

It is well known that the natural morphism

$$\mathbb{C}[\mathrm{Re}\mathfrak{a}_M] \otimes A(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_Z \to A(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$$

defined by $(Q,\phi) \mapsto \big(g \mapsto Q(\log_M(m_P(g))\big) \cdot \phi(g)$ is an isomorphism, using the special structure of $A_{M(\mathbb{A})}$-finite functions and the Fourier analysis over the compact space $A_{M(\mathbb{A})}\backslash Z_{M(\mathbb{A})}$. Consequently, we also obtain a natural isomorphism

$$\mathbb{C}[\mathrm{Re}\mathfrak{a}_M] \otimes A_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_Z \to A_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\xi.$$

Set also $\Pi_0(M(\mathbb{A}))_\xi$ be isomorphism classes of irreducible representations of $M(\mathbb{A})$ occurring in the space $A_0(M(F)\backslash M(\mathbb{A}))_\xi$, and

$$\Pi_0(M(\mathbb{A})) := \cup_{\xi \in \mathrm{Hom}(Z_{M(\mathbb{A})},\mathbb{C}^*)} \Pi_0(M(\mathbb{A}))_\xi.$$

For any $\pi \in \Pi_0(M(\mathbb{A}))_\xi$, set $A_0(M(F)\backslash M(\mathbb{A}))_\pi$ to be the isotypic component of type π of $A_0(M(F)\backslash M(\mathbb{A}))_\xi$, i.e, the set of cusp forms of $M(\mathbb{A})$ generating a semi-simple isotypic $M(\mathbb{A})$-module of type π. Set

$$\begin{aligned} &A_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\pi \\ &\quad := \{\phi \in A_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A})) : \phi_k \in A_0(M(F)\backslash M(\mathbb{A}))_\pi, \forall k \in \mathbb{K}\}. \end{aligned}$$

Clearly

$$A_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\xi = \oplus_{\pi \in \Pi_0(M(\mathbb{A}))_\xi} A_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\pi.$$

More generally, let $V \subset A(M(F)\backslash M(\mathbb{A}))$ be an irreducible $M(\mathbb{A})$-module with π_0 the induced representation of $M(\mathbb{A})$. Then we call π_0 an automorphic representation of $M(\mathbb{A})$. Denote by $A(M(F)\backslash M(\mathbb{A}))_{\pi_0}$ the isotypic subquotient module of type π_0 of $A(M(F)\backslash M(\mathbb{A}))$. Then

$$V \otimes \mathrm{Hom}_{M(\mathbb{A})}(V, A(M(F)\backslash M(\mathbb{A}))) \simeq A(M(F)\backslash M(\mathbb{A}))_{\pi_0}.$$

Set

$$\begin{aligned}&A(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_{\pi_0}\\ &\qquad := \{\phi \in A(U(\mathbb{A})M(F)\backslash G(\mathbb{A})) : \phi_k \in A(M(F)\backslash M(\mathbb{A}))_{\pi_0}, \forall k \in \mathbb{K}\}.\end{aligned}$$

Moreover if $A(M(F)\backslash M(\mathbb{A}))_{\pi_0} \subset A_0(M(F)\backslash M(\mathbb{A}))$, we call π_0 a cuspidal representation.

Two automorphic representations π and π_0 of $M(\mathbb{A})$ are said to be equivalent if there exists $\lambda \in X_M^G$ such that $\pi \simeq \pi_0 \otimes \lambda$. This, in practice, means that $A(M(F)\backslash M(\mathbb{A}))_\pi = \lambda \cdot A(M(F)\backslash M(\mathbb{A}))_{\pi_0}$. That is, for any $\phi_\pi \in A(M(F)\backslash M(\mathbb{A}))_\pi$ there exists a $\phi_{\pi_0} \in A(M(F)\backslash M(\mathbb{A}))_{\pi_0}$ such that $\phi_\pi(m) = m^\lambda \cdot \phi_{\pi_0}(m)$. Consequently,

$$A(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\pi = (\lambda \circ m_P) \cdot A(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_{\pi_0}.$$

Denote by $\mathfrak{P} := [\pi_0]$ the equivalence class of π_0. Then $\mathfrak{P}$ is an X_M^G-principal homogeneous space, hence admits a natural complex structure. Usually we call $(M, \mathfrak{P})$ a cuspidal datum of G if π_0 is cuspidal. Also for $\pi \in \mathfrak{P}$ set $\mathrm{Re}\,\pi := \mathrm{Re}\,\chi_\pi = |\chi_\pi| \in \mathrm{Re}X_M$, where χ_π is the central character of π, and $\mathrm{Im}\,\pi := \pi \otimes (-\mathrm{Re}\,\pi)$.

Now fix an irreducible automorphic representation π of $M(\mathbb{A})$ and $\phi \in A(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\pi$, define the associated Eisenstein series $E(\phi, \pi) : G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ by

$$E(\phi, \pi)(g) := \sum_{\delta \in P(F)\backslash G(F)} \phi(\delta g).$$

Then there is an open cone $\mathcal{C} \subset \mathrm{Re}X_M^G$ such that if $\mathrm{Re}\,\pi \in \mathcal{C}$, $E(\lambda \cdot \phi, \pi \otimes \lambda)(g)$ converges uniformly for g in a compact subset of $G(\mathbb{A})$ and λ in an open neighborhood of 0 in X_M^G. For example, if $\mathfrak{P} = [\pi]$ is cuspidal, we may even take $\mathcal{C}$ to be the cone $\{\lambda \in \mathrm{Re}X_M^G : \langle \lambda - \rho_P, \alpha^\vee \rangle > 0, \forall \alpha \in \Delta_P^G\}$. As a direct consequence, $E(\phi, \pi) \in A(G(F)\backslash G(\mathbb{A}))$. That is, Eisenstein series $E(\phi, \pi)$ are automorphic forms.

As noticed above, being an automorphic form, $E(\phi, \pi)$ is of moderate growth. However, in general it is not integrable over $Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})$. To remedy this, classically, as initiated in the so-called Rankin-Selberg method, analytic truncation is used. From Fourier analysis, we understand that the problematic terms are the so-called constant terms, which are of moderate growth, so by cutting off them, the reminding one is rapidly increasing and hence integrable.

In general, it is very difficult to make such an analytic truncation intrinsically related with arithmetic properties of number fields. (See however, [Z] and [Ar1,2].) On the other hand, Eisenstein series themselves are quite intrinsic arithmetical invariants. Thus it is natural for us on one hand to keep Eisenstein series unchanged while on the other to find new moduli spaces, which themselves intrinsically parametrize certain modular objects, and over which Eisenstein series are integrable.

This is exactly what we are doing now. As said, we are going to view Eisenstein series as something globally defined, and use a geometric truncation for the space $G(F)\backslash G(\mathbb{A})$ so that the integrations of the Eisenstein series over the newly obtained moduli spaces give us naturally non-abelian L functions for function fields.

As such, let us now come back to the group $G = GL_r$, then as in 3.1, we obtain the moduli space $\mathcal{M}_{F,r}^{\leq p}$ and hence a well-defined integration

$$L_{F,r}^{\leq p}(\phi,\pi) := \int_{\mathcal{M}_{F,r}^{\leq p}} E(\phi,\pi)(g)\,dg, \qquad \mathrm{Re}\,\pi \in \mathcal{C}.$$

3.3. New non-abelian *L*-functions. However, in such a general form, we do not know whether the latest defined integration has any nice properties (such as meromorphic continuation and functional equations, etc.). It is to remedy this that we make a further selection about automorphic forms.

Fix then a convex polygon $p : [0, r] \to \mathbb{R}$ as in II.2 so as to obtain the moduli space $\mathcal{M}_{F,r}^{\leq p}$. Set $G = GL_r$, fix the minimal parabolic subgroup P_0 corresponding to the partition $(1, \cdots, 1)$ with M_0 consisting of diagonal matrices. Fix a standard parabolic subgroup $P_I = U_I M_I$ corresponding to the partition $I = (r_1, \ldots, r_{|P|})$ of r with M_I the standard Levi and U_I the unipotent radical.

Then for a fixed irreducible automorphic representation π of $M_I(\mathbb{A})$, choose

$$\phi \in A(U_I(\mathbb{A})M_I(F)\backslash G(\mathbb{A}))_\pi \cap L^2(U_I(\mathbb{A})M_I(F)\backslash G(\mathbb{A})) := A^2(U_I(\mathbb{A})M_I(F)\backslash G(\mathbb{A}))_\pi,$$

where $L^2(U_I(\mathbb{A})M_I(F)\backslash G(\mathbb{A}))$ denotes the space of L^2 functions on the space $Z_{G(\mathbb{A})}U_I(\mathbb{A})M_I(F)\backslash G(\mathbb{A})$. Denote the associated Eisenstein series by $E(\phi,\pi) \in A(G(F)\backslash G(\mathbb{A}))$.

Definition. A rank r non-abelian L-function $L_{F,r}^{\leq p}(\phi,\pi)$ for the function field F associated to an L^2-automorphic form $\phi \in A^2(U_I(\mathbb{A})M_I(F)\backslash G(\mathbb{A}))_\pi$ is defined by the following integration

$$L_{F,r}^{\leq p}(\phi,\pi) := \int_{\mathcal{M}_{F,r}^{\leq p}} E(\phi,\pi)(g)\,dg, \qquad \mathrm{Re}\,\pi \in \mathcal{C}.$$

More generally, for any standard parabolic subgroup $P_J = U_J M_J \supset P_I$ (so that the partition J is a refinement of I), we have the corresponding relative

Eisenstein series

$$E_I^J(\phi,\pi)(g) := \sum_{\delta\in P_I(F)\backslash P_J(F)} \phi(\delta g), \qquad \forall g \in P_J(F)\backslash G(\mathbb{A}).$$

It is well known that there is an open cone $\mathcal{C}_I^J$ in $\mathrm{Re}X_{M_I}^{P_J}$ such that for $\mathrm{Re}\,\pi \in \mathcal{C}_I^J$, $E_I^J(\phi,\pi) \in A(P_J(F)\backslash G(\mathbb{A}))$. Here $X_{M_I}^{P_J}$ is defined similarly as X_M^G with G replaced by P_J. Then we have a well-defined relative non-abelian L-function

$$L_{F,r}^{P_J;\leq p}(\phi,\pi) := \int_{\mathcal{M}_{F,r}^{P_J;\leq p}} E_I^J(\phi,\pi)(g)\,dg, \qquad \mathrm{Re}\ \pi \in \mathcal{C}_I^J.$$

Remarks. (1) Here when defining non-abelian L-functions we assume that ϕ comes from a single irreducible automorphic representations, but this restriction is rather artificial and can be removed easily: We add such a restriction only for the purpose of giving the constructions and results in a very neat way.

(2) We point out that the following discussion for non-abelian L-functions holds for relative non-abelian L-functions as well, with certain simple modifications in a well-known manner.

3.4. Meromorphic extension, rationality and functional equations. With the same notation as above, set $\mathfrak{P} = [\pi]$. For $w \in W$, the Weyl group of $G = GL_r$, fix once and for all representative $w \in G(F)$ of w. Set $M' := wMw^{-1}$ and denote the associated parabolic subgroup by $P' = U'M'$. W acts naturally on automorphic representations, from which we obtain an equivalence classes $w\mathfrak{P}$ of automorphic representations of $M'(\mathbb{A})$. As usual, define the associated intertwining operator $M(w,\pi)$ by

$$(M(w,\pi)\phi)(g) := \int_{U'(F)\cap wU(F)w^{-1}\backslash U'(\mathbb{A})} \phi(w^{-1}n'g)dn', \qquad \forall g \in G(\mathbb{A}).$$

One checks that if $\langle \mathrm{Re}\pi, \alpha^\vee\rangle \gg 0, \forall \alpha \in \Delta_P^G$:

(i) for a fixed ϕ, $M(w,\pi)\phi$ depends only on the double coset $M'(F)wM(F)$. So $M(w,\pi)\phi$ is well-defined for $w \in W$;

(ii) the above integral converges absolutely and uniformly for g varying in a compact subset of $G(\mathbb{A})$;

(iii) $M(w,\pi)\phi \in A(U'(\mathbb{A})M'(F)\backslash G(\mathbb{A}))_{w\pi}$; and if ϕ is L^2, which from now on we always assume, so is $M(w,\pi)\phi$.

Basic facts of non-abelian L-functions. With the same notation above:

(I) **Meromorphic Continuation.** $L_{F,r}^{\leq p}(\phi,\pi)$ for $\mathrm{Re}\,\pi \in \mathcal{C}$ is well-defined and admits a unique meromorphic continuation to the whole space $\mathfrak{P}$;

(II) **Rationality.** $L_{F,r}^{\leq p}(\phi,\pi)$ for $\mathrm{Re}\,\pi\in\mathcal{C}$ is a rational function on $\mathfrak{P}$;

(III) **Functional Equations.** As meromorphic functions on $\mathfrak{P}$,

$$L_{F,r}^{\leq p}(\phi,\pi)=L_{F,r}^{\leq p}(M(w,\pi)\phi,w\pi),\qquad \forall w\in W.$$

Proof. This is a direct consequence of the fundamental results of Langlands and Morris on Eisenstein series and spectrum decompositions. (See e.g, [Mor1,2], [La] and/or [MW]). Indeed, if ϕ is cuspidal, by definition, (I) is a direct consequence of Prop. II.15, Thm. IV.1.8 of [MW], (II) is a direct consequence of Thm. IV.1.11 of [MW], (II) is a direct consequence of the proof of Thm. IV.1.12 of [MW] and (II) is a direct consequence of Thm. IV.1.10 of [MW].

More generally, if ϕ is only L^2, then by Langlands and Morris' theory of Eisenstein series and spectral decomposition, ϕ may be obtained as the residue of relative Eisenstein series coming from cusp forms, since ϕ is L^2 automorphic. As such then (I), (II) and (II) are direct consequences of the proof of VI.2.1(i) at p. 264 of [MW].

3.5. Holomorphicity and singularities. Let $\pi\in\mathfrak{P}=[\pi]$ and $\alpha\in\Delta_M^G$ and $\alpha\in R^+(T_M,G)$. Denote by $n(\alpha)$ the smallest integer $n>0$ such that $\alpha^{*n\lambda}=1$ for all $\lambda\in\mathrm{Fix}_{X_M^G}(\mathfrak{P}):=\{\nu\in X_M^G:\pi\otimes\nu=\nu\}$ with α^* as defined at p.16-17 of [MW]. Define then the function $h:\mathfrak{P}\to\mathbb{C}$ by $\pi\otimes\lambda\mapsto\alpha^{*n(\alpha)\lambda}-1$ for all $\lambda\in X_M^G\simeq\mathfrak{a}_M^G$. Set $H:=\{\pi'\in\mathfrak{P}:h(\pi')=0\}$ and call it a root hyperplane. Clearly the function h is determined by H, hence we also denote h by h_H. Note also that root hyperplanes depend on the base point π we choose.

Let D be a set of root hyperplanes. Then:

(i) the singularities of a meromorphic function f on $\mathfrak{P}$ is said to be carried out by D if for all $\pi\in\mathfrak{P}$, there exist $n_\pi:D\to\mathbb{Z}_{\geq 0}$ zero almost everywhere such that $\pi'\mapsto\left(\Pi_{H\in D}h_H(\pi')^{n_\pi(H)}\right)\cdot f(\pi')$ is holomorphic at π';

(ii) the singularities of f are said to be without multiplicity at π if $n_\pi\in\{0,1\}$;

(iii) D is said to be locally finite, if for any compact subset $C\subset\mathfrak{P}$, $\{H\in D:H\cap C\neq\emptyset\}$ is finite.

Basic facts of non-abelian L-functions. With the same notation above:

(IV) **Holomorphicity.** (i)When $\mathrm{Re}\,\pi\in\mathcal{C}$, $L_{F,r}^{\leq p}(\phi,\pi)$ is holomorphic;

(ii) $L_{F,r}^{\leq p}(\phi,\pi)$ is holomorphic at π where $\mathrm{Re}\,\pi=0$;

(V) **Singularities.** (i) There is a locally finite set of root hyperplanes D such that the singularities of $L_{F,r}^{\leq p}(\phi,\pi)$ are carried out by D;

(ii) The singularities of $L_{F,r}^{\leq p}(\phi,\pi)$ are without multiplicities at π if $\langle\mathrm{Re}\pi,\alpha^\vee\rangle\geq 0,\forall\alpha\in\Delta_M^G$;

(iii) There are only finitely many of singular hyperplanes of $L_{F,r}^{\leq p}(\phi,\pi)$ which intersect $\{\pi\in\mathfrak{P}:\langle\mathrm{Re}\,\pi,\alpha^\vee\rangle\geq 0,\forall\alpha\in\Delta_M\}$.

Proof. As above, this is a direct consequence of the fundamental results of Langlands and Morris on Eisenstein series and spectrum decompositions. (See e.g, [Mor1,2], [La] and/or [MW]). Indeed, if ϕ is a cusp form, (IV.i) is a direct consequence of Lemma IV.1.7 of [MW], while (IV.ii) and (IV) are direct consequence of Prop. IV.1.11 of [MW].

In general when ϕ is only L^2 automorphic, then we have to use the theory of Langlands and Morris to realize ϕ as the residue of relative Eisenstein series defined using cusp forms. (See e.g., item (5) at p. 198 and the second half part of p. 232 of [MW].)

As such, (IV) and (V) are direct consequence of the definition of residue datum and the compatibility between residue and Eisenstein series as stated for example under item (3) at p. 263 of [MW].

Chapter II.4. A closed formula for the Abelian part.

4.1. Modified analytic truncation. Let $G = GL_r$ and $P_0 = M_0U_0$ be the minimal parabolic subgroup corresponding to the partition $(1, \dots, 1)$. Let $P_1 = M_1U_1$ be a fixed standard parabolic subgroup with M_1 the standard Levi and U_1 the unipotent radical.

For a function field F with $\mathbb{A}$ the ring of adeles, let π be an irreducible automorphic representation of $M_1(\mathbb{A})$. Denote by $A^2(U_1(\mathbb{A})M_1(F)\backslash G(\mathbb{A}))_\pi$ the space of L^2-automorphic forms in the isotypic component $A(U_1(\mathbb{A})M_1(F)\backslash G(\mathbb{A}))_\pi$.

Then for a fixed convex polygon $p : [0, r] \to \mathbb{Q}$ and any L^2-automorphic form $\phi \in A^2(U_1(\mathbb{A})M_1(F)\backslash G(\mathbb{A}))_\pi$ we have the associated non-abelian L-function

$$L_{F,r}^{\leq p}(\phi; \pi) := \int_{\mathcal{M}_{F,r}^{\leq p}} E(\phi, \pi)(g) \cdot d\mu(g), \qquad \mathrm{Re}\,\pi \in \mathcal{C}$$

where $E(\phi, \pi)$ denotes the Eisenstein series associated to ϕ and $\mathcal{C} \subset X_{M_1}^G$ is a certain positive cone in 3.3 over which Eisenstein series $E(\phi, \pi)$ converges. Recall that in 3.4, we showed that $L_{F,r}^{\leq p}(\phi; \pi)$ admits a meromorphic continuation to the whole space $\mathfrak{P} := [\pi]$, the $X_{M_1}^G$ homogeneous space consisting of automorphic representations equivalent to π whose typical element is $\pi \otimes \lambda$ with $\lambda \in X_{M_1}^G$.

On the other hand, for a suitably regular $T \in \mathrm{Re}\mathfrak{a}_M^*$, following Arthur, (see [Ar1] and [OW],) we have the analytic truncation $\Lambda^T f$ for any continuous function f on $Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})$ defined by

$$(\Lambda^T f)(g) := \sum_P (-1)^{\dim(A_P/Z_G)} \sum_{\delta \in P(F)\backslash G(F)} f_P(\delta g) \cdot \hat{\tau}_P(\log_M m_P(\delta g) - T).$$

(For unknown notation, which is commonly used in Arthur's theory, please see [Ar1,2] and [OW].) Apply this analytic truncation to the constant function 1, by Prop 1.1 of [Ar1], we obtain a characteristic function for a certain compact subset in $Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})$, which we denote by $\Lambda^T\left(Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})\right)$. Thus, for $\phi \in A^2(M(F)U(\mathbb{A})\backslash G(\mathbb{A}))$, we have a well-defined integration

$$L_{F,r}^T(\phi,\pi) := \int_{\Lambda^T\left(Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})\right)} E(\phi,\pi)(g)\cdot dg, \qquad \mathrm{Re}\pi \in \mathcal{C}.$$

Moreover, it is well known that for analytic truncations,

$$\Lambda^T \circ \Lambda^T = \Lambda^T$$

based on the following miracle—By Lemma 1.1 of [Ar2], the constant term of $\Lambda^T\phi(x)$ along with any standard parabolic subgroup P_1 is zero unless $\varpi(H_0(x)-T)<0$ for all $\varpi \in \hat{\Delta}_1$. As a direct consequence,

$$\begin{aligned} L_{F,r}^T(\phi,\pi) &= \int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})} \Lambda^T 1(g)\cdot E(\phi,\pi)(g)\cdot dg \\ &= \int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})} (\Lambda^T\circ\Lambda^T)1(g)\cdot E(\phi,\pi)(g)\cdot dg \\ &= \int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})} \Lambda^T 1(g)\cdot \Lambda^T E(\phi,\pi)(g)\cdot dg \end{aligned}$$

since Λ^T is self-adjoint. But this latest integration is simply

$$\int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})} 1(g)\cdot(\Lambda^T\circ\Lambda^T)E(\phi,\pi)(g)\cdot dg$$

since $\Lambda^T E(\phi,\pi)$ is rapidly decreasing and 1 is of moderate growth. That is to say,

$$L_{F,r}^T(\phi,\pi) = \int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})} \Lambda^T E(\phi,\pi)(g)\cdot dg.$$

One may try to apply such a discussion to geometric truncations as well. For this, attach to a fixed concave polygon $p:[0,r]\to\mathbb{R}$ with the property $p(0)=p(r)=0$ an element $T_p=(t_p^1,\cdots,t_p^r)\in\mathfrak{a}_0$ by the conditions

$$\lambda_i(T_p)=t_p^i-t_p^{i+1} := [p(i)-p(i-1)]-[p(i+1)-p(i)]>0, i=1,2,\ldots,r-1.$$

Here as usual $\{\lambda_i=e_i-e_{i+1}\}_{i=1}^{r-1}$ denotes the collection of positive roots of GL_r. Then one checks (see [We2] for details) that:

(i) T_p is in the positive cone of $\mathfrak{a}_0$; and

(ii) $\tau_P(-H(g)-T_P)=1 \Leftrightarrow p_P^g \triangleright_P p$.

Note in particular that in (ii), τ_P instead of $\hat{\tau}_P$ is used. In other words, positive chambers rather than positive cones are used in geometric truncation. We should also point out that this discussion is motivated by Lafforgue [Laf].

Moreover, following Lafforgue [Laf], introduce a modified truncation with respect to a polygon p by

$$(\Lambda_p f)(g) := \sum_P (-1)^{\dim(A_P/Z_G)} \sum_{\delta\in P(F)\backslash G(F)} f_P(\delta g)\cdot \mathbf{1}(p_P^{\delta g} >_P p).$$

Denote thus obtained moduli space (from $\Lambda_p 1$) by $\Lambda_p(Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A}))$. Then essentially, the compact space $\Lambda_p(Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A}))$ is our moduli space $\mathcal{M}_{F,r}^{\leq p}$ by Prop. II.2 and (ii) above. In this way, our problem becomes to study

$$\mathcal{L}_{F,r}^{\leq p}(\phi;\pi) := \int_{\Lambda_p(Z_{G(\mathbb{A})}F(F)\backslash G(\mathbb{A}))} E(\phi,\pi)\cdot d\mu(g), \qquad \mathrm{Re}\ \pi\in\mathcal{C}.$$

4.2. A close formula when ϕ is a cusp form. For general ϕ, this turns to be a very challenging problem. Our aim here is to see what happens for $L_{F,r}^{\leq p}(\phi;\pi)$ when ϕ is a cusp form. Motivated by the result of Langlands-Arthur on the inner product of truncated Eisenstein series [Ar1,2] (see also [OW]), we go as follows:

We begin with a formula for the truncated Eisenstein series. This then leads to the consideration of constant terms of Eisenstein series. While it is difficult to precisely describe constant terms of Eisenstein series $E(\phi,\pi)$ associated with general automorphic form ϕ, it becomes rather easy when ϕ is cuspidal. Indeed, for $\phi\in A_0(U_1(\mathbb{A})M_1(F)\backslash G(\mathbb{A}))$ and a fixed standard parabolic subgroup $P=MU$, it is well known that

$$E_P(\phi,\pi)(g) = \sum_{w\in W(M_1,M)}\ \sum_{m\in M(F)\cap wP_1(F)w^{-1}\backslash M(F)} (M(w,\pi)\phi(\pi))\,(mg),$$

where $W(M_1,M)$ consisting of element $w\in W$ such that wM_1w^{-1} is a standard Levi of M and $w^{-1}(\beta)>0$ for all $\beta\in R^+(T_0,M)$ and $R^+(T_0,M)$ denotes the set of positive roots related to (T_0,M).

Therefore,

$$\begin{aligned}\Lambda_p E(\phi,\pi) &= \sum_P (-1)^{\dim A_P/Z_G} \sum_{\delta\in P(F)\backslash G(F)} E_P(\phi,\pi)(\delta g)\cdot \mathbf{1}(p_P^{\delta g} >_P p)\\ &= \sum_P (-1)^{\dim A_P/Z_G} \sum_{\delta\in P(F)\backslash G(F)}\\ &\qquad \sum_{w\in W(M_1,M)}\ \sum_{\xi\in M(F)\cap wP_1(F)w^{-1}\backslash M(F)} (M(w,\pi)\phi)\,(\xi\delta g)\cdot \mathbf{1}(p_P^{\delta g} >_P p).\end{aligned}$$

Now for any standard parabolic subgroup P_2, set $W(\mathfrak{a}_1,\mathfrak{a}_2)$ to be the set of distinct isomorphisms from $\mathfrak{a}_1$ onto $\mathfrak{a}_2$ obtained by restricting elements in W to $\mathfrak{a}_1$, where $\mathfrak{a}_i$ denotes $\mathfrak{a}_{P_i}, i=1,2$ Then one checks by definition easily that $W(M_1,M)$ is a union over all P_2 of elements $w\in W(\mathfrak{a}_1,\mathfrak{a}_2)$ such that (i) $w\mathfrak{a}_1=\mathfrak{a}_2\supset\mathfrak{a}_P$; and (ii) $w^{-1}(\alpha)>0, \forall\alpha\in\Delta_{P_2}^P$.

Hence,

$$\begin{aligned}
&\Lambda_p E(\phi,\pi)\\
&= \sum_{P_2} \sum_{w\in W(\mathfrak{a}_1,\mathfrak{a}_2), P\supset P_2, w^{-1}(\alpha)>0, \forall\alpha\in\Delta_{P_2}^P} (-1)^{\dim A_P/Z_G}\\
&\qquad \sum_{\delta\in P(F)\backslash G(F)} 1(p_P^{\delta g} >_P p)\cdot \sum_{\xi\in M(F)\cap wP_1(F)w^{-1}\backslash M(F)} (M(w,\pi)\phi)\,(\xi\delta g)\\
&= \sum_{P_2} \sum_{w\in W(\mathfrak{a}_1,\mathfrak{a}_2)} (-1)^{\dim A_{P_w}/Z_G} \sum_{\{P:P_2\subset P\subset P_w, w^{-1}(\alpha)>0,\forall\alpha\in\Delta_{P_2}^P\}} (-1)^{\dim A_P/A_{P_w}}\\
&\qquad \sum_{\delta\in P(F)\backslash G(F)} 1(p_P^{\delta g} >_P p)\cdot \sum_{\xi\in M(F)\cap wP_1(F)w^{-1}\backslash M(F)} (M(w,\pi)\phi)\,(\xi\delta g).
\end{aligned}$$

where for a given w, we define $P_w \supset P$ by the conidition that

$$\Delta_{P_2}^{P_w} = \{\alpha\in\Delta_{P_2} : (w\pi)(\alpha^\vee) > 0\}.$$

Therefore, since

$$1(p_P^{\xi\delta g} >_P p) = 1(p_P^{\delta g} >_P p), \qquad \forall\delta\in P(F)\backslash G(F), \xi\in P_2(F)\backslash P(F),$$

we have

$$\begin{aligned}
&\Lambda_p E(\phi,\pi)\\
&= \sum_{P_2} \sum_{w\in W(\mathfrak{a}_1,\mathfrak{a}_2)} (-1)^{\dim A_{P_w}/Z_G} \sum_{\{P:P_2\subset P\subset P_w, w^{-1}(\alpha)>0,\forall\alpha\in\Delta_{P_2}^P\}} (-1)^{\dim A_P/A_{P_w}}\\
&\qquad \sum_{\delta\in P(F)\backslash G(F)} \sum_{\delta\in P_2(F)\backslash P(F)} \Big(1(p_P^{\xi\delta g} >_P p)\cdot (M(w,\pi)\phi)\,(\xi\delta g)\Big)\\
&= \sum_{P_2} \sum_{w\in W(\mathfrak{a}_1,\mathfrak{a}_2)} (-1)^{\dim A_{P_w}/Z_G} \sum_{\{P:P_2\subset P\subset P_w, w^{-1}(\alpha)>0,\forall\alpha\in\Delta_{P_2}^P\}} (-1)^{\dim A_P/A_{P_w}}\\
&\qquad \sum_{\delta\in P_2(F)\backslash G(F)} \Big(1(p_P^{\delta g} >_P p)\cdot (M(w,\pi)\phi)\,(\delta g)\Big)\\
&= \sum_{P_2} \sum_{\delta\in P_2(F)\backslash G(F)} \sum_{w\in W(\mathfrak{a}_1,\mathfrak{a}_2)} (-1)^{\dim A_{P_w}/Z_G} (M(w,\pi)\phi)\,(\xi\delta g)\\
&\qquad\qquad \sum_{\{P:P_2\subset P\subset P_w, w^{-1}(\alpha)>0,\forall\alpha\in\Delta_{P_2}^P\}} (-1)^{\dim A_P/A_{P_w}} 1(p_P^{\delta g} >_P p).
\end{aligned}$$

Set now

$$1(P_2;p;w) := \sum_{\{P:P_2\subset P\subset P_w, w^{-1}(\alpha)>0,\forall\alpha\in\Delta_{P_2}^P\}} (-1)^{\dim A_P/Z_G} 1(p_P^{\delta g} >_P p).$$

Then, we obtain the following:

LEMMA. *With the same notation as above,*

$$\Lambda_p E(\phi,\pi)(g) = \sum_{P=MU}\sum_{\delta\in P(F)\backslash G(F)}\sum_{w\in W(M_1), wM_1w^{-1}=M} (M(w,\pi)\phi)(\delta g)\cdot 1(P;p;w)(\delta g).$$

In the following calculation, we will pay no attention to the convergence: One may justify our discussion using either the standard method in [Ar1], and/or [OW], to first create a rapid decreasing function via pseudo-Eisenstein series or the same wave packets, then apply the inversion formula, or regularized integrations in [JLR]. Also if Λ_p were idempotent, we would have had no chance to get an essential non-abelian part in our non-abelian L-function.

With these comments in mind, now we introduce what we call the abelian part $L_{F,r}^{\leq p,\mathrm{ab}}$ of our non-abelian L function $L_{F,r}^{\leq p}$ by setting

$$L_{F,r}^{\leq p,\mathrm{ab}}(\phi,\pi) := \int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})} \Lambda_p E(\phi,\pi)(g)\, d\mu(g).$$

If Λ_p were idempotent, we would have had no chance to get an essential non-abelian part in our non-abelian L-function. It is this abelian part which we are going to calculate.

At it stands,

$$L_{F,r}^{\leq p,\mathrm{ab}}(\phi,\pi) = \int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})} \sum_{P=MU}\sum_{\delta\in P(F)\backslash G(F)}\sum_{w\in W(M_1), wM_1w^{-1}=M} (M(w,\pi)\phi)(\delta g) \cdot 1(P;p;w)(\delta g)\, dg.$$

From an unfolding trick, it is simply

$$\begin{aligned}&\sum_P \sum_{w\in W(M_1), wM_1w^{-1}=M} \int_{Z_{G(\mathbb{A})}P(F)\backslash G(\mathbb{A})} (1(P_2;p;w)(g)\cdot (M(w,\pi)\phi)(g))\, dg\\ &\quad = \sum_P \sum_{w\in W(M_1), wM_1w^{-1}=M} \int_{Z_{G(\mathbb{A})}U(\mathbb{A})M(F)\backslash G(\mathbb{A})} (1_P(P;p;w)(g)\cdot (M(w,\pi)\phi)(g))\, dg,\end{aligned}$$

where as usual $1_P(P;p;w)(g) := \int_{U(F)\backslash U(\mathbb{A})} 1(P;p;w)(ng)\, dn$ denotes the constant term of $1(P;p;w)(g)$ along P.

To evaluate this latest integral, we decompose it into a double integrations over

$$\begin{aligned}&\Big(Z_{G(\mathbb{A})}(Z_M(F)\cap Z_{M(\mathbb{A})}\backslash Z_{G(\mathbb{A})}\cdot Z^1_{M(\mathbb{A})}\Big)\times\Big(Z_{G(\mathbb{A})}Z^1_{M(\mathbb{A})}U(\mathbb{A})M(F)\backslash G(\mathbb{A})\Big)\\ &\quad=\Big(Z_{G(\mathbb{A})}\cdot Z^1_{M(\mathbb{A})}\backslash Z_{M(\mathbb{A})}\Big)\times\big(Z_{M(\mathbb{A})}U(\mathbb{A})M(F)\backslash G(\mathbb{A})\big),\end{aligned}$$

where $Z^1_{M(\mathbb{A})}=Z_{M(\mathbb{A})}\cap M(\mathbb{A})^1$. That is to say,

$$\begin{aligned}L_{F,r}^{\leq p,\mathrm{ab}}(\phi,\pi)=\sum_{P=MU}\sum_{w\in W(M_1),wM_1w^{-1}=M}&\int_{Z_{M(\mathbb{A})}U(\mathbb{A})M(F)\backslash G(\mathbb{A})}dg\\ &\cdot\int_{Z_{G(\mathbb{A})}\cdot Z^1_{M(\mathbb{A})}\backslash Z_{M(\mathbb{A})}}(1_P(P;p;w)(zg)\cdot(M(w,\pi)\phi)(zg))\,dz.\end{aligned}$$

Note now that since $X^G_{M_1}$ has no torsion, there exists a unique element π_0 of $\mathfrak{P}:=[\pi]$ whose restriction to $A^G_{M_1(\mathbb{A})}$ is trivial. This then allows to canonically identified $X^G_{M_1}$ with $\mathfrak{P}$ via $\lambda_\pi\in X^G_{M_1}\mapsto\pi:=\pi_0\otimes\lambda_\pi\in\mathfrak{P}$. Hence without loss of generality, we may simply assume that the restriction of π to $A^G_{M_1(\mathbb{A})}$ is trivial.

Therefore,

$$\begin{aligned}L_{F,r}^{\leq p,\mathrm{ab}}(\phi,\pi)=\sum_{P=MU}\sum_{w\in W(M_1),wM_1w^{-1}=M}&\int_{Z_{M(\mathbb{A})}U(\mathbb{A})M(F)\backslash G(\mathbb{A})}(M(w,\pi)\phi)(g)\,dg\\ &\cdot\int_{Z_{G(\mathbb{A})}\cdot Z^1_{M(\mathbb{A})}\backslash Z_{M(\mathbb{A})}}(1_P(P;p;w)(zg)\cdot)\,z^{\rho_P+w\pi}dz.\end{aligned}$$

However as g may be chosen in $G(\mathbb{A})^1$, clearly, the integration

$$\int_{Z_{G(\mathbb{A})}\cdot Z^1_{M(\mathbb{A})}\backslash Z_{M(\mathbb{A})}}(1_P(P;p;w)(zg)\cdot)\,z^{\rho_P+w\pi}dz$$

is independent of g. Denote it by $W(P;p;w;\pi)$. As a direct consequence, we obtain the following:

A CLOSED FORMULA. *With the same notation as above, for $\phi\in A_0(U_1(\mathbb{A})M_1(F)\backslash G(\mathbb{A}))_\pi$,*

$$L_{F,r}^{\leq p,ab}(\phi;\pi)=\sum_{P=MU}\sum_{w\in W(M_1),wM_1w^{-1}=M}\big(W(P;p;w;\pi)\cdot\langle M(w,\pi)\phi,1\rangle\big).$$

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