Module of coinvariants and a generalization of Poincaré–Hopf theorem

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§1 Module of coinvariants

- G : gp $\curvearrowright$ M: ab. gp.

$\longrightarrow$ $M_G = M/\{m - gm \in M \mid m \in M, g \in G\}$: module of coinvariants.

$= \mathbb{Z} \otimes_{\mathbb{Z}[G]} M$

$= H_0(G; M)$

- We are interested in $\ell^\infty(G, \mathbb{R})_G$ $\left(\begin{array}{l} \ell^\infty(G, \mathbb{R}) = \{(a_g)_g \mid a_g \in \mathbb{R}, \sup_g |a_g| < \infty\} \\ h \cdot (a_g)_g = (a_{h^{-1}g})_g \end{array} \right)$

<u>Ex</u>

If G is finite, $\ell^\infty(G, \mathbb{R})_G \cong \mathbb{R}$. $\square$

<u>Thm</u>

$\mathbb{1}$: const. seq. with value 1.

G: gp. TFAE.

(1) G is amenable

(2) $[\mathbb{1}] \neq 0$ in $\ell^\infty(G, \mathbb{R})_G$

(3) $\ell^\infty(G, \mathbb{R})_G \neq 0$ $\square$

<u>Ex.</u>

$G = \mathbb{Z}$. Consider $\ell^\infty(G, \mathbb{Z})_G$.

<u>Fact.</u> $K_0(C^*_u(|\mathbb{Z}|)) \cong \ell^\infty(\mathbb{Z}, \mathbb{Z})_G$ (follows from Pimsner–Voiculescu exact seq.)

<u>Fact.</u> $\ell^\infty(\mathbb{Z}, \mathbb{Z})_{\mathbb{Z}}$ is a $\mathbb{Q}$-vect. sp. of uncountable dimension. $\square$

<u>Ex</u>

$\frac{1}{2}[\mathbb{1}] = [\ldots, 1, 0, 1, 0, 1, 0, \ldots]$

$\because$

$[\ldots, 1, 0, 1, 0, \ldots] = [\ldots, 0, 1, 0, 1, \ldots]$

$[\ldots, 1, 0, 1, 0, \ldots] + [\ldots, 0, 1, 0, 1, \ldots] = [\mathbb{1}]$. ∎

§2 Integration.

- M : G-covering over a closed oriented n-mfd. M/G.
  - Fix a fundamental domain $K \subset M$.
  - Fix a Riem met on M/G and lift it to M.
  - $\Omega^q_b M = \{\omega \in \Omega^q M \mid \sup_x |\omega_x| < \infty, \sup_x |(d\omega)_x| < \infty\}$: bdd. q-forms

$\rightsquigarrow$ $\Omega^*_b M$, $H^*_b M$: bdd de Rham cpx. and cohomology.

2025.12.04
Kagoshima

- Define $\int_M : H_b^n M \to \ell^\infty(G,\mathbb{R})_G$
$$[\omega] \longmapsto \left[\left(\int_{gK}\omega\right)_g\right]$$

Lem. $\int_M$ is well-defined □

Lem. (Milizia) $\int_M$ is isom. and independent of choice of K. □

Cor. G: nonamenable $\Rightarrow$ $H_b^n M = 0$ □

Rem.

When G : fin., this is the usual integration through $\ell^\infty(G,\mathbb{R})_G \cong \mathbb{R}$ □

§3 Poincaré–Hopf thm.

- $G \curvearrowright M$ as before, $\pi : M \to M/G$: proj.
  - $u \in \Omega_{vc}^n T(M/G)$: Thom class of $T(M/G)$ (tangent bdl.) with vertically cpt. support
    $\hat{u} := \pi^* u \in \Omega_{vc,b}^n TM$: bounded Thom class.
  - $v : M \to TM$: vect field
    v is bdd. $\overset{\text{def}}{\Longleftrightarrow}$ $\sup_x |v_x| < \infty$ and $\sup_x |\nabla v_x| < \infty$.
    v is tame $\overset{\text{def}}{\Longleftrightarrow}$ (i) $\exists \delta > 0$ s.t. $x \neq y$, $v_x = 0, v_y = 0 \Rightarrow d(x,y) \geq \delta$.
    (ii) $\exists \delta' > 0$, $\exists \varepsilon > 0$ s.t. $d(x, v^{-1}(0)) \geq \delta \Rightarrow |v_x| \geq \varepsilon$.
  - v is bdd. $\Rightarrow$ $v^* \hat{u} \in \Omega_b^n M$.
    Define $\hat{e}(TM) = [v^* \hat{u}] \in H_b^n M$ (independent of v)
    $\hat{\chi}(M) = \int_M \hat{e}(TM) \in \ell^\infty(G,\mathbb{R})_G$.

Lem.

$\hat{\chi}(M) = \chi(M/G)\,[\mathbb{1}]$ in $\ell^\infty(G,\mathbb{R})_G$. □

Thm (bdd. Poincaré–Hopf thm)

v : bdd. & tame then (if any zero is contained in some $\mathrm{Int}\, gK$)
$$\left[\left(\sum_{x \in v^{-1}(0) \cap gK} \mathrm{ind}_x v\right)_g\right] = \chi(M/G)\,[\mathbb{1}] \quad \text{in } \ell^\infty(G,\mathbb{R}).$$

In particular, if G : amenable and $\chi(M/G) \neq 0$,
then any bdd. tame vect. field on M has ∞ ly many zeros. □

(Fact G : fin. gen. ∞ gp., $(a_g)_g$: $a_g = 0$ for all but finitely many g.
then $[(a_g)_g] = 0$ in $\ell^\infty(G,\mathbb{R})_G$)

2025.12.04
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§4 Further development

- Morse inequality on M (G-covering over a non-cpt closed mfd. M/G)
  - # of crit. pts is considered to be an element of $l^\infty(G,\mathbb{R})_G$
  - The proof is given by a method of heat kernel.
- Singular theory
  - We define the cohomology theory $H^*_{ub}(X)$ called the <u>uniform bounded cohomology</u>.
  - We establish an obstruction theory similar to the classical one for uniformly continuous maps.
  - As a corollary, we obtain uniform versions of Poincaré-Hopf thm and Lefschetz-Hopf thm (for fixed pts of self maps)
    - In particular, the converse of today's main thm. holds:
      If $\chi(M/G)[\mathbb{1}] = 0$ in $l^\infty(G,\mathbb{R})$, then any bdd. vector field on M can be deformed into a nowhere zero one "boundedly".
      In particular, if M is not amenable, then any vect. field on M can be deformed into a nowhere zero one boundedly.