Higher homotopy normalities in topological groups

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Algebraic Topology and Related Fields

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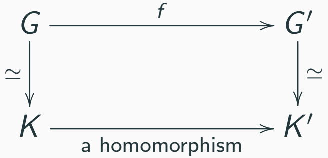
1. Higher homotopy associativity and commutativity
    - See how multiplicative structures and classifying spaces are related.
2. Higher homotopy normality
    - Recall classical homotopy normal maps, which are generalizations of normal subgroups (and crossed modules).
    - Define higher homotopy variant of homotopy normal maps, called $N_k(\ell)$-map $(0 \leq k, \ell \leq \infty)$.
3. Results
    - The main theorem characterizes $N_k(\ell)$-maps by a method of fiberwise homotopy theory.
    - Some computational examples on classical groups.

Higher homotopy associativity and commutativity

H-map

- A map $f\colon G \to G'$ between topological groups is said to be an H-map if
$$f \circ \mu \simeq \mu \circ (f \times f).$$
We can say an H-map is "a homomorphism up to homotopy".
- However, H-map is far from homomorphism. There exists an H-map $f\colon G \to G'$ not homotopy equivalent to any homomorphism $f'\colon K \to K'$ between topological groups as in the following diagram:



- This difference can be understood by considering higher homotopy.

Classifying space and projective spaces

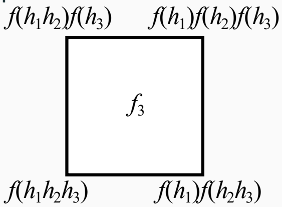
- The classifying space BG of a topological group G is constructed as the quotient
$$BG = \left(\coprod_{i \geq 0} \Delta^i \times G^i \right) \Big/ \sim$$
by some simplicial relation $\sim$.
- The image of $\Delta^k \times G^k$ is written by $B_k G$ (k-th projective space). Then we obtain the filtration
$$* = B_0 G \subset \Sigma G = B_1 G \subset B_2 G \subset \cdots \subset B_k G \subset \cdots \subset BG.$$
- If $f \colon G \to G'$ is a homomorphism, then we have the induced maps
$$B_k f \colon B_k G \to B_k G', \quad Bf \colon BG \to BG'.$$

Examples of classifying spaces and projective spaces

- When $G = S^0, S^1$ and S^3, $B_k S^0 = \mathbb{R}P^k$, $B_k S^1 = \mathbb{C}P^k$ and $B_k S^3 = \mathbb{H}P^k$, respectively.
- When $G = \mathrm{U}(n)$, $B\,\mathrm{U}(n) \simeq G_n(\mathbb{C}^\infty)$ (the Grassmannian of n-planes in $\mathbb{C}^\infty$). In general, $B_k\,\mathrm{U}(n)$ is not a manifold.

A_∞-map

- A map $f\colon G \to G'$ is said to be an A_∞-map if it admits an A_∞-form $\{f_i\colon I^{i-1} \times G^i \to G'\}_{i\geq 1}$, which describes how the associativity is preserved through f.
- What is an A_∞-form $\{f_i\colon I^{i-1} \times G^i \to G'\}_{i\geq 1}$?
  - $f_1 = f$.
  - $f_2\colon I \times G^2 \to G'$ is a homotopy between $f \circ \mu$ and $\mu \circ (f \times f)$.
  - $f_3\colon [0,1]^2 \times G^3 \to G'$ is depicted as follows.



- We will call a pair $(f, \{f_i\}_i)$ an A_∞-map.

Classifying space and A_∞-map

- **Theorem (Sugawara, 1960).** A map $f\colon G \to G'$ admits an A_∞-form if and only if the suspension $\Sigma f\colon \Sigma G \to \Sigma G'$ extends to a map between the classifying space $BG \to BG'$:

$$\begin{array}{ccc} \Sigma G & \xrightarrow{\Sigma f} & \Sigma G' \\ {\scriptstyle \text{inclusion}}\big\downarrow & & \big\downarrow{\scriptstyle \text{inclusion}} \\ BG & \underset{\exists}{\dashrightarrow} & BG' \end{array}$$

- By the simplicial loop group construction, an A_∞-map $f\colon G \to G'$ is homotopy equivalent to some homomorphism between topological groups in the previous sense.

A_k-map

- Stasheff (1963) considered the intermediate objects between H-map and A_∞-map: a map $f\colon G \to G'$ is said to be an A_k-map if it admits an A_k-form $\{f_i\colon I^{i-1} \times G^i \to G'\}_{1\le i\le k}$.
  - An A_1-map is just a map.
  - An A_2-map is an H-map (with homotopy $f \circ \mu \simeq \mu \circ (f \times f)$).
- **Theorem (Stasheff, 1963).** A map $f\colon G \to G'$ admits an A_k-form if and only if the suspension $\Sigma f\colon \Sigma G \to \Sigma G$ extends to a map from $B_k G$ to BG':

$$
\begin{array}{ccc}
\Sigma G & \xrightarrow{f} & \Sigma G' \\
{\scriptstyle \text{inclusion}}\downarrow & & \downarrow{\scriptstyle \text{inclusion}} \\
B_k G & \underset{\exists}{\dashrightarrow} & BG'
\end{array}
$$

Homotopy commutativity

- A topological group G is said to be homotopy commutative if the Samelson product
$$G \wedge G \to G, \qquad (x, y) \mapsto xyx^{-1}y^{-1}$$
is null-homotopic.
- Through the isomorphisms
$$[G \wedge G, G] \cong [G \wedge G, \Omega BG] \cong [\Sigma G \wedge G, BG],$$
the Samelson product corresponds to the Whitehead product $[\iota, \iota]$ of the inclusion $\iota\colon \Sigma G \to BG$.
  - So, G is homotopy commutative if and only if $[\iota, \iota] = 0$.

Higher homotopy commutativity

- A topological group G is said to be a C_k-space in the sense of Sugawara (defined by McGibbon 1989) if the multiplication $G \times G \to G$ is an A_k-map.
  - Remark. This definition is similar to the fact that a group G is abelian if and only if the multiplication $G \times G \to G$ is a homomorphism.
  - G is a C_2-space if and only if G is homotopy commutative.
- An equivalent condition is as follows: the wedge sum of the inclusion

$$B_k G \vee B_k G \to BG$$

extends over the union

$$\bigcup_{i+j=k} B_i G \times B_j G \to BG.$$

Higher homotopy commutativity (continued)

- Remark. There is another notion of C_k-space in the sense of Williams, which is a bit weaker than Sugawara's.
- Remark. G is a C_∞-space in the sense of Sugawara if and only if BG is an H-space. This condition is much weaker than requiring G to be a double loop space (equivalently, BG to be a loop space).
- The higher homotopy commutativity of Lie groups and their p-localizations has been extensively studied. Roughly speaking, the p-local homotopy commutativity gets higher as p gets bigger. Let us see a typical argument to show the non-commutativity in the next slide.

Example of non-commutativity

Let $G = \mathrm{SU}(2) = S^3$ and p an odd prime. Suppose $k \geq \frac{p+1}{2}$ and the wedge sum of the inclusion

$$\mathbb{H}P^k \vee \mathbb{H}P^k \to \mathbb{H}P^\infty$$

extends to a map

$$f\colon B = \bigcup_{i+j=k} \mathbb{H}P^i \times \mathbb{H}P^j \to \mathbb{H}P^\infty.$$

We know $\mathcal{P}^1 x = a x^{\frac{p+1}{2}}$ with $a \neq 0$ for a generator $x \in H^4(\mathbb{H}P^\infty; \mathbb{F}_p)$. Then the coefficient of $x^i \times x^j$ with $i, j > 0$ in $f^* \mathcal{P}^1 x$ is nontrivial in

$$H^*(B; \mathbb{F}_p) = \mathbb{F}_p[x \times 1, 1 \times x]/(x^i \times x^j \mid i + j > k).$$

But this contradicts to the computation of $\mathcal{P}^1 f^* x$ by the Cartan formula and $f^* x = x \times 1 + 1 \times x$. This contradicts to $\mathcal{P}^1 x = a x^{\frac{p+1}{2}}$ and $a \neq 0$. Therefore, $\mathrm{SU}(2)$ is not p-locally a C_k-space.

Higher homotopy normality

Crossed module

- In the rest of this talk, let H and G be topological groups of homotopy types of CW complexes.
- Recall that a normal subgroup $H \subset G$ is a subgroup stable under the inner automorphisms.
- Crossed module is a generalization of normal subgroup to general homomorphisms $H \to G$.
- **Definition.** A (topological) crossed module consists of homomorphisms $f\colon H \to G$ and $\rho\colon G \to \mathrm{Aut}(H)$ satisfying the conditions
  - $\rho(f(h))(x) = hxh^{-1}$ for any $x, h \in H$,
  - $f(\rho(g)(x)) = gf(x)g^{-1}$ for any $x \in H$ and $g \in G$.
- Remark. $f(\rho(g)(x)) = gf(x)g^{-1} \Leftrightarrow g^{-1}f(\rho(g)(x))g = f(x)$.

Homotopy quotient of crossed module

- **Theorem (Farjoun–Segev, 2010).** The Borel construction $K = EH \times_H G$ of a crossed module $f\colon H \to G$ naturally inherits a group structure. Moreover, there exists a homotopy fiber sequence

$$\cdots \to H \xrightarrow{f} G \to K \to BH \xrightarrow{Bf} BG \to BK.$$

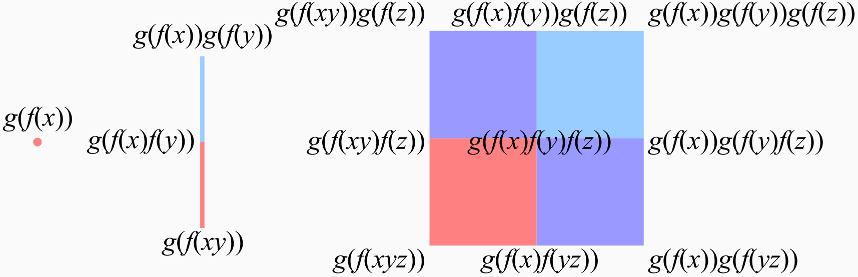
- When $H \subset G$ is a closed normal subgroup, the natural homotopy equivalence $K \to G/H$ is a homomorphism. Then we should consider that K is "the homotopy quotient group of a homotopically normal subgroup".
- My initial motivation for higher homotopy normality was to generalize this result to higher homotopy theoretic setting. However, $N_\infty(\infty)$-map turned out to be much weaker than crossed module (this kind of phenomena will appear later).

Topological category $\mathcal{A}_k$

- Let us give a naive construction of a category of topological groups and A_k-maps between them. Our argument could work in other appropriate higher categorical setting.
- Let
$$\mathcal{A}_k(G, G') \subset \prod_{1 \leq i \leq k} \mathrm{Map}(I^{i-1} \times G^i, G')$$
be the space of A_k-maps.

Topological category $\mathcal{A}_k$ (continued)

- We have the composition of A_k-maps as follows:



Modifying $\mathcal{A}_k(G, G')$ and the composition like Moore path, we can make this composition unital and associative.

Topological category $\mathcal{A}_k$ (continued)

- Then we obtain the topological category $\mathcal{A}_k$ of topological groups and A_k-maps.
  - In particular, the space of self A_k-maps $\mathcal{A}_k(G, G)$ is a topological monoid.
- We have a continuous functor $B_k \colon \mathcal{A}_k \to \mathbf{Spaces}_*$.
- **Theorem (T. 2016).** The following composite is a weak homotopy equivalence:

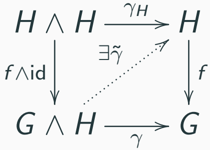
$$\mathcal{A}_k(G, G') \xrightarrow{B_k} \mathrm{Map}_*(B_k G, B_k G') \xrightarrow{\text{inclusion}} \mathrm{Map}_*(B_k G, BG').$$

$N_k(\ell)$-map

- Let $\mathrm{conj}_H \colon H \to \mathcal{A}_\ell(H,H)$ denote the conjugation $\mathrm{conj}_H(h)(x) = hxh^{-1}$.
- **Definition (T. 2023).** A homomorphism $f \colon H \to G$ is an $N_k(\ell)$-map if an A_k-map $\rho \colon G \to \mathcal{A}_\ell(H,H)$ is given and the following conditions hold:
  - $\rho \circ f$ is homotopic to conj_H as an A_ℓ-map,
  - the map $* \to \mathcal{A}_\ell(H,G)$, $* \mapsto f$ is A_k-equivariant with respect to the action of G,
  - the higher homotopies appearing in the first and second conditions coincide on H.
- This is a higher homotopy analogue of crossed module.

$N_1(1)$-map and James' homotopy normal map

- **Definition (McCarty 1964).** A homomorphism $f\colon H \to G$ is homotopy normal (an $N_1(1)$-map) if there exists a map $\tilde{\gamma}\colon G \wedge H \to H$ making the diagram



  commute up to homotopy and the homotopies comapatible with the stationary homotopy of the outer square.
- Homotopy normal map in the sense of James (1967) only requires the commutativity of the lower triangle.

Immediate consequences

- If $f\colon H \to G$ is an $N_k(\ell)$-map and $k \geq k'$ and $\ell \geq \ell'$, then f is an $N_{k'}(\ell')$-map.
- If $f\colon H \to G$ is a crossed module, then f is an $N_\infty(\infty)$-map.
- The homomorphism $f\colon H \to *$ is an $N_k(\ell)$-map if and only if $\mathrm{conj}_H\colon H \to \mathcal{A}_\ell(H, H)$ is homotopic to the constant map as an A_k-map.
  - The latter condition is equivalent to being a $C(k, \ell)$-space introduced by Kishimoto and Kono (2010).
  - $C(\infty, \infty)$-space and Sugawara C_∞-space are known to be equivalent. Then we conclude that $H \to *$ is an $N_\infty(\infty)$-map if and only if BH is an H-space.
  - This is analogous to the fact that $H \to *$ is a crossed module if and only if H is commutative.

Results

Equivariant and fiberwise homotopy theory

- The Borel construction defines the correspondence

$$\text{a } G\text{-space } X \quad \mapsto \quad \text{a fiberwise space } EG \times_G X \to BG.$$

  This provides an "equivalence" between the G-equivariant homotopy theory and the fiberwise homotopy theory over BG in an appropriate sense.
  - EG denotes the universal G-bundle over BG. The restriction to $B_k G$ will be denoted by $E_k G$.
- The idea of the main theorem is based on this kind of equivalence.

Main theorem

- **Theorem (T. 2023).** Let $f\colon H \to G$ be a homomorphism and $F\colon E_k H \times_H H \to E_k G \times_G G$ denote the induced map of f. Then f is an $N_k(\ell)$-map if and only if there exists a fiberwise A_ℓ-space $\mathcal{E} \to B_k G$ and F factors as
$$E_k H \times_H H \xrightarrow{\phi} \mathcal{E} \xrightarrow{\psi} E_k G \times_G G$$
up to homotopy over $B_k f\colon B_k H \to B_k G$ such that the following conditions hold:
  - ϕ covers $B_k f$ and ψ covers the identity on $B_k G$,
  - ϕ and ψ are fiberwise A_ℓ-maps,
  - ϕ is a weak homotopy equivalence on each fiber,
  - the restriction of $\psi \circ \phi$ to the fiber over the basepoint is homotopic to f as an A_ℓ-map.

Remark on main theorem

- Roughly, this theorem states that $f\colon H \to G$ is an $N_k(\ell)$-map if and only if the following "unusual" factorization of $F\colon E_kH \times_H H \to E_kG \times_G G$ exists:

$$
\begin{array}{ccccc}
H & = & H & \xrightarrow{f} & G \\
\downarrow & & \downarrow & & \downarrow \\
E_kH \times_H H & \longrightarrow & \mathcal{E} & \longrightarrow & E_kG \times_G G \\
\downarrow & & \downarrow & & \downarrow \\
B_kH & \xrightarrow[B_kf]{} & B_kG & = & B_kG
\end{array}
$$

Remark on main theorem (continued)

- The "usual" factorization is as follows. The middle column is induced from the conjugation action of H on G through f.

$$
\begin{array}{ccccc}
H & \xrightarrow{f} & G & = & G \\
\downarrow & & \downarrow & & \downarrow \\
E_k H \times_H H & \longrightarrow & E_k H \times_H G & \longrightarrow & E_k G \times_G G \\
\downarrow & & \downarrow & & \downarrow \\
B_k H & = & B_k H & \xrightarrow[B_k f]{} & B_k G
\end{array}
$$

- This factorization is possible for any homomorphism f.

H-structure on Borel construction

- **Theorem (T. 2023).** Let $f\colon H \to G$ be a homomorphism. Then the Borel construction $X = EH \times_H G$ is an H-space if f is an $N_k(k)$-map and $\operatorname{cat} X \leq k$ (the naturality of the H-structure is unknown).
- **Example.** Let $H = K(\mathbb{Q}, 2n-1)$ and $G = K(\mathbb{Q}, 4n-1)$. Consider the homomorphism $f\colon H \to G$ with classifying map $Bf\colon K(\mathbb{Q}, 2n) \to K(\mathbb{Q}, 4n)$ corresponding to $u^2 \in H^{4n}(K(\mathbb{Q}, 2n); \mathbb{Q})$. Then the Borel construction is

$$EH \times_H G \simeq \operatorname{hofib}(Bf) \simeq S^{2n}_{(0)}.$$

Since $S^{2n}_{(0)}$ does not admit an H-structure and $\operatorname{cat} S^{2n}_{(0)} = 1$, f is not an $N_1(1)$-map (a map is not necessarily homotopy normal even if its target is an ∞-loop space!).

Preceding results on examples

- There have been many results on homotopy normality of Lie groups.
- (James 1967)
  The inclusion $\mathrm{U}(m) \to \mathrm{U}(n)$ is not (2-locally) homotopy normal in the sense of James for $1 \leq m < n$. Similar results hold for $\mathrm{O}(m) \to \mathrm{O}(n)$ $(2 \leq m < n)$ and $\mathrm{Sp}(m) \to \mathrm{Sp}(n)$ for $1 \leq m < n$.
- Other results include: McCarty (1964), James (1971), Kachi (1982), Furukawa (1985), Furukawa (1987), Furukawa (1995), Kudou–Yagita (1998), Kudou–Yagita (2003), Kono–Nishimura (2003), Nishimura (2006), Kishimoto–T. (2018).
- These results suggest that $H \to G$ tends to fail to be p-locally homotopy normal for small prime p.

Higher homotopy normality of $\mathrm{SU}(m) \to \mathrm{SU}(n)$

- Applying the fiberwise projective space functor, the main theorem provides an obstruction theory for $N_k(\ell)$-map.
- By a typical argument using Steenrod operations as mentioned before for commutativity, we obtain the following result.

- **Theorem (T. 2023).**
  - If $p \geq kn + \ell m$, then the inclusion $\mathrm{SU}(m) \to \mathrm{SU}(n)$ is a p-local $N_k(\ell)$-map.
  - If $\max\{kn-2, (k-1)n+2\} < p \leq kn + 2(\ell - 1)$, then the inclusion $\mathrm{SU}(2) \to \mathrm{SU}(n)$ is not a p-local $N_k(\ell)$-map for $n \geq 3$.
  - If $\max\{kn-m, (k-1)n+2\} < p \leq kn + (\ell - 2)m$, then the inclusion $\mathrm{SU}(m) \to \mathrm{SU}(n)$ is not a p-local $N_k(\ell)$-map for $2 \leq m < n$.
- This result is not very sharp. For example, the normality is not determined when $kn + (\ell - 2)m < p < kn + \ell m$.
- A similar result is obtained for $\mathrm{SO}(2m+1) \to \mathrm{SO}(2n+1)$.

3-local normality of SU(2) → SU(3)

k	1	2	3	4	5
$N_k(1)$	✗	✗	✗	✗	✗
$N_k(2)$	✗	✗	✗	✗	✗
$N_k(3)$	✗	✗	✗	✗	✗
$N_k(4)$	✗	✗	✗	✗	✗
$N_k(5)$	✗	✗	✗	✗	✗

5-local normality of SU(2) → SU(3)

k	1	2	3	4	5
$N_k(1)$	✓	?	?	?	?
$N_k(2)$	✗	✗	✗	✗	✗
$N_k(3)$	✗	✗	✗	✗	✗
$N_k(4)$	✗	✗	✗	✗	✗
$N_k(5)$	✗	✗	✗	✗	✗

7-local normality of SU(2) → SU(3)

k	1	2	3	4	5
$N_k(1)$	✓	?	?	?	?
$N_k(2)$	✓	✗	✗	✗	✗
$N_k(3)$	✗	✗	✗	✗	✗
$N_k(4)$	✗	✗	✗	✗	✗
$N_k(5)$	✗	✗	✗	✗	✗

11-**local normality of** SU(2) → SU(3)

k	1	2	3	4	5
$N_k(1)$	✓	✓	✓	?	?
$N_k(2)$	✓	✓	✗	✗	✗
$N_k(3)$	✓	?	✗	✗	✗
$N_k(4)$	✓	✗	✗	✗	✗
$N_k(5)$	✗	✗	✗	✗	✗

Summary

- $N_k(\ell)$-map is a higher homotopical analogue of crossed module and normal subgroup.
- $N_k(\ell)$-map is characterized by fiberwise A_ℓ-maps over k-th projective spaces.
- The Borel construction $EH \times_H G$ of an $N_k(k)$-map $f\colon H \to G$ is an H-space if $\operatorname{cat} EH \times_H G \leq k$ holds.
- Fiberwise projective space provides a method to detect obstructions to being $N_k(\ell)$-maps.

Thank you!