Limit distributions for the discretization error of stochastic Volterra equations

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The discretization of the stochastic differential equations (SDE) has been studied by many researchers for many years. Consider a standard SDE and its discretization:

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s, \quad t \in [0,T],$$
$$\hat{X}_t = X_0 + \int_0^t b(\hat{X}_{\frac{[ns]}{n}})\mathrm{d}s + \int_0^t \sigma(\hat{X}_{\frac{[ns]}{n}})\mathrm{d}W_s, \quad t \in [0,T],$$

where b and σ are continuously differentiable functions whose derivatives being bounded and uniformly continuous. In the 1990s, the limit distribution of the scaled error $U^{(n)} = n^{-1/2}(X - \hat{X})$ was studied by Jacod, Kurtz and Protter; see Kurtz and Protter [3] or Jacod and Protter [2]. They proved that $(W, U^{(n)})$ converges in law to (W, U) as n tends to infinity, where U is the solution of the following SDE:

$$U_t = \int_0^t U_s(b'(X_s)\mathrm{d}s + \sigma'(X_s)\mathrm{d}W_s) - \frac{1}{\sqrt{2}}\int_0^t \sigma'(X_s)\sigma(X_s)\mathrm{d}\hat{W}_s$$

with $\hat{W}$ being a standard Brownian motion independent of W.

The aim of our study is to extend their result to stochastic Volterra equations (SVE) which are represented by

$$X_t = X_0 + \int_0^t K(t-s)b(X_s)\mathrm{d}s + \int_0^t K(t-s)\sigma(X_s)\mathrm{d}W_s, \quad t \in [0,T], \tag{1}$$

where $K(t) = \frac{t^{H-1/2}}{\Gamma(H+1/2)}$, $H \in (0, 1/2)$. Likewise in the case of SDEs, let $\hat{X}$ be the solution of the discretized SVE of (1), that is,

$$\hat{X}_t = X_0 + \int_0^t K(t-s)b(\hat{X}_{\frac{[ns]}{n}})\mathrm{d}s + \int_0^t K(t-s)\sigma(\hat{X}_{\frac{[ns]}{n}})\mathrm{d}W_s, \quad t \in [0,T].$$

We denote by C_0^0 the set of the continuous functions on $[0,T]$ vanishing at $t=0$ and by $C_0^{0,\lambda}$ the set of the λ-Hölder continuous functions with the same property. Then the following theorem is proved, which is our main result.

Theorem 1. *Let $\epsilon \in (0,H)$. Then the process $U^{(n)} = n^H(X - \hat{X})$ stably converges in law in $C_0^{0,H-\epsilon}$ to a process U which is the unique continuous solution of the SVE*

$$U_t = \int_0^t K(t-s)U_s(b'(X_s)\mathrm{d}s + \sigma'(X_s)\mathrm{d}W_s) - \frac{1}{\sqrt{\Gamma(2H+2)\sin \pi H}}\int_0^t K(t-s)\sigma'(X_s)\sigma(X_s)\mathrm{d}\hat{W}_s, \tag{2}$$

where $\hat{W}$ is a standard Brownian motion independent of W.

To prove Theorem 1, we start with the following decomposition:

$$U_t^{(n)} \approx \int_0^t K(t-s)U_s^{(n)}(b'(\hat{X}_{\frac{[ns]}{n}})\mathrm{d}s + \sigma'(\hat{X}_{\frac{[ns]}{n}})\mathrm{d}W_s) + \int_0^t K(t-s)b'(\hat{X}_{\frac{[ns]}{n}})\mathrm{d}\langle V^{(n)}, W\rangle_s + \int_0^t K(t-s)\mathrm{d}\tilde{V}_s^{(n)}, \tag{3}$$

where $U^{(n)}, V^{(n)}$, and $\tilde{V}^{(n)}$ are defined as

$$U^{(n)} = n^H(X - \hat{X}), \quad V^{(n)} = n^H\int_0^{\cdot}(\hat{X}_s - \hat{X}_{\frac{[ns]}{n}})\mathrm{d}W_s, \quad \tilde{V}^{(n)} = \int_0^{\cdot}\sigma'(\hat{X}_{\frac{[ns]}{n}})\mathrm{d}V_s^{(n)}.$$

Remark that the difference between both sides of (3) converges to zero in $C_0^{0,H-\epsilon}$ for any $\epsilon \in (0,H)$ as n goes to infinity. We have the limits of the quadratic variation and covariation of $V^{(n)}$ and W.

Lemma 2. *For all $t \in [0,T]$,*

$$\langle V^{(n)}, V^{(n)} \rangle_t \xrightarrow[\text{in } L_1]{n\to\infty} \frac{1}{\Gamma(2H+2)\sin\pi H} \int_0^t \sigma(X_s)^2 \mathrm{d}s,$$
$$\langle V^{(n)}, W \rangle_t \xrightarrow[\text{in } L_1]{n\to\infty} 0.$$

Then Lemma 2 and the results of Jacod [1] lead us to specify the limit distribution of $V^{(n)}$ in Lemma 3.

Lemma 3. *The process $V^{(n)}$ stably converges in law in C_0^0 to a continuous process V of the following form:*

$$V = \frac{1}{\sqrt{\Gamma(2H+2)\sin\pi H}} \int_0^\cdot \sigma(X_s)\mathrm{d}\hat{W}_s,$$

where $\hat{W}$ is a standard Brownian motion independent of W, namely, $\langle W, \hat{W} \rangle = 0$.

The convergence of $\tilde{V}^{(n)}$ will be considered in Lemma 4. Some outcomes of Kurtz and Protter [4], the general results for the convergence of stochastic integrals in the Skorokhod topology, are used here.

Lemma 4. *The process $\tilde{V}^{(n)}$ stably converges in law in C_0^0 to the process*

$$\tilde{V} = \int_0^\cdot \sigma'(X_s)\mathrm{d}V_s.$$

Regarding the stochastic integral of the fractional kernel K with respect to the process $\tilde{V}^{(n)}$ as a Riemann-Liouville fractional derivative, we can use the continuity of the derivative from $C_0^{0,\lambda}$ into $C_0^{0,\lambda-(H-1/2)}$ for $\lambda > 1/2 - H$ as in Lemma 7; see Samko, Kilbas, and Marichev [5]. We additionally construct a useful criterion of tightness in $C_0^{0,\lambda}$ as in Theorem 8. The weak convergence in C_0^0 of $V^{(n)}$ is obtained in Lemma 3, and then, Lemma 4 and Corollary 9 imply that the last term of (3) converges in law in the appropriate space. Moreover, we can show that the second integral of (3) tends weakly to zero in $C^{0,H-\epsilon}$. As a consequence, the tightness of $U^{(n)}$ in $C_0^{0,H-\epsilon}$ is obtained, and thus, together with the uniqueness of the solution U of (2), the desired weak convergence is shown.

Definition 5. Let $f \in C_0^{0,\lambda}$ and $\alpha = 1/2 - H$. We define the integral operator $\mathcal{J}^\alpha$ as

$$(\mathcal{J}^\alpha f)(t) := K(t)f(t) - \int_0^t (f(t) - f(s))K'(t-s)\mathrm{d}s.$$

Proposition 6. *Let Y be a process such that the stochastic integral $\int_0^t K(t-s)\mathrm{d}Y_s$ is well defined. Then the integral is almost surely represented by $\mathcal{J}^\alpha$ as*

$$\int_0^t K(t-s)\mathrm{d}Y_s = (\mathcal{J}^\alpha Y)(t).$$

Lemma 7 (Samko, Kilbas, and Marichev [5])**.** *The operator $\mathcal{J}^\alpha$ is bounded (continuous) from $C_0^{0,\lambda}$ into $C_0^{0,\lambda-\alpha}$.*

Theorem 8. *Let $\{Y^{(n)}\}_{n\in\mathbb{N}}$ be a sequence of $C_0^{0,\alpha}$-valued random variables. If $\mathbb{E}[\|Y^{(n)}\|_{C_0^{0,\alpha}}]$ is bounded uniformly in n, the sequence $\{Y^{(n)}\}_{n\in\mathbb{N}}$ is tight in $C_0^{0,\beta}$ for $0 < \beta < \alpha$.*

Corollary 9. *Let $Y^{(n)}$ be a stochastic process which converges to a process Y weakly in C_0^0 as n goes to infinity. If $Y^{(n)}$ satisfies $\mathbb{E}[\|Y^{(n)}\|_{C_0^{0,\alpha}}] \le C$ for some C uniformly in n, it converges to V weakly in $C_0^{0,\beta}$ for any positive $\beta < \alpha$.*

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