A partial rough path space for rough volatility

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A rough volatility model is a stochastic volatility model for an asset price process with volatility being rough, meaning that the Hölder regularity of the volatility path is less than half. We focus on a model of the following form:

$$dS_t = \sigma(S_t) f(\hat{X}_t) dX_t, \quad S_0 \in \mathbb{R} \tag{0.1}$$

where X is a Brownian motion and $\hat{X}$ is a Riemann-Liouville fractional Brownian motion with the Hurst parameter $H < 1/4$. Our aim is to develop a rough path framework to deal with such an (Itô) integration of an uncontrolled path. Suppose that $x : [0,T] \to \mathbb{R}^d$ ($d \geqq 1$), $\hat{x} : [0,T] \to \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ are good enough. By Taylor expansion, for $\tau_1 < \tau_2$ (these are close enough),

$$\int_{\tau_1}^{\tau_2} f(\hat{x}_r) \otimes dx_r \sim f(\hat{x}_{\tau_1}) \otimes (x_{\tau_2} - x_{\tau_1}) + \sum_{i=1}^{n} \frac{1}{i!} \nabla^i f(\hat{x}_{\tau_1}) \int_{\tau_1}^{\tau_2} (\hat{x}_r - \hat{x}_{\tau_1})^i \otimes dx_r$$

$$\int_{\tau_1}^{\tau_2} \left(\int_{\tau_1}^{r} dy_u \right) dy_r \sim \sum_{0 \leqq j+k \leqq n} \frac{1}{j!k!} \nabla^j f(\hat{x}_{\tau_1}) \nabla^k f(\hat{x}_{\tau_1}) \int_{\tau_1}^{\tau_2} (\hat{x}_r - \hat{x}_{\tau_1})^k \left(\int_{\tau_1}^{r} (\hat{x}_u - \hat{x}_{\tau_1})^j \otimes dx_u \right) \otimes dx_r,$$

where $y_t := \int_0^t f(\hat{x}_r) \otimes dx_r$. Therefore, if we preliminarily could define

$$Z^{(i)}_{st} := \frac{1}{i!} \int_s^t (\hat{x}_r - \hat{x}_s)^i \otimes dx_r, \quad A^{(j,k)}_{st} := \frac{1}{k!} \int_s^t (\hat{x}_r - \hat{x}_s)^k Z^{(j)}_{sr} \otimes dx_r,$$

we would be able to define a rough path integral $\int f(\hat{x}_r) \otimes dx_r$. By the linearity of the integration and the binomial theorem, $Z^{(i)}$ and $A^{(j,k)}$ satisfy the following formula respectively;
for any $i, j, k \geqq 0$, and $s \leqq u \leqq t$,

$$Z^{(i)}_{st} = Z^{(i)}_{su} + \sum_{p=0}^{i} \frac{1}{(i-p)!} (\hat{X}_{su})^{i-p} Z^{(p)}_{ut} \tag{0.2}$$

$$A^{(j,k)}_{st} = A^{(j,k)}_{su} + \sum_{q=0}^{k} \frac{1}{(k-q)!} (\hat{X}_{su})^{k-q} Z^{(j)}_{su} \otimes Z^{(q)}_{ut} + \sum_{p=0}^{j} \sum_{q=0}^{k} \frac{1}{(j-p)!(k-q)!} (\hat{X}_{su})^{j+k-p-q} A^{(p,q)}_{ut} \tag{0.3}$$

In light of these formulas, we define the following space $\Omega_{(\alpha,H)\text{-Hld}}$. In this article, fix $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, $H \in (0, \frac{1}{2})$, $\Delta_T := \{(s,t); 0 \leqq s \leqq t \leqq T\}$, $I := \{i \in \mathbb{Z}; iH + \alpha \leqq 1\}$, and $J := \{(j,k) \in \mathbb{Z} \times \mathbb{Z}; (j+k)H + 2\alpha \leqq 1\}$.

Definition 0.1. An (α, H) rough path $\mathbb{X} = \left(\hat{X}, Z^{(i)}, A^{(j,k)}\right)_{i \in I, (j,k) \in J}$ is a triplet of functions on Δ_T satisfying the following conditions; for any $i \in I$, $(j,k) \in J$ and $s \leqq u \leqq t$,
(i) $\hat{X}$ is $\mathbb{R}$ -valued, $Z^{(i)}$ is $\mathbb{R}^d$ -valued, and $A^{(j,k)}$ is $\mathbb{R}^d \otimes \mathbb{R}^d$ -valued function.
(ii) *Modified Chen's relation*; $\hat{X}_{st} = \hat{X}_{su} + \hat{X}_{ut}$, and $Z^{(i)}$ and $A^{(j,k)}$ satisfy (0.2) and (0.3) respectively.
(iii) *Hölder regularities*;

$$|\hat{X}_{st}| \lesssim |t-s|^H, \quad |Z^{(i)}_{st}| \lesssim |t-s|^{iH+\alpha}, \quad |A^{(j,k)}_{st}| \lesssim |t-s|^{(j+k)H+2\alpha}$$

Let $\Omega_{(\alpha,H)\text{-Hld}}$ denote the space of the (α, H) - rough paths. We define a metric function $d_{(\alpha,H)}$ on $\Omega_{(\alpha,H)\text{-Hld}}$ and $|||\mathbb{X}|||_{(\alpha,H)}$ as following;

$$d_{(\alpha,H)}(\mathbb{X}, \mathbb{Y}) := ||\hat{X} - \hat{Y}||_{H\text{-Hld}} + \sum_{i \in I, (j,k) \in J} ||Z(\mathbb{X})^{(i)} - Z(\mathbb{Y})^{(i)}||_{iH+\alpha\text{-Hld}} + ||A(\mathbb{X})^{(j,k)} - A(\mathbb{Y})^{(j,k)}||_{(j+k)H+2\alpha\text{-Hld}}$$

$$|||\mathbb{X}|||_{(\alpha,H)} := ||\hat{X}||_{H\text{-Hld}} + \sum_{i \in I, (j,k) \in J} ||Z^i||_{iH+\alpha\text{-Hld}} + ||A^{(j,k)}||_{(j+k)H+2\alpha\text{-Hld}}$$

Definition 0.2. Fix $\mathbb{X} \in \Omega_{(\alpha,H)\text{-Hld}}$. We define $Y^{(1)}$ and $Y^{(2)}$ as follows if exist;

$$Y^{(1)}_{st} := \lim_{|\mathcal{P}|\searrow 0} \sum_{p=1}^{N} \sum_{i\in I} \nabla^i f(\hat{x}_{t_{p-1}}) Z^{(i)}_{t_{p-1}t_p}, \quad Y^{(2)}_{st} := \lim_{|\mathcal{P}|\searrow 0} \sum_{p=1}^{N} \left(Y^{(1)}_{t_0 t_{p-1}} \otimes Y^{(1)}_{t_{p-1}t_p} + \sum_{(j,k)\in J} \nabla^j f(\hat{x}_{t_{p-1}}) \nabla^k f(\hat{x}_{t_{p-1}}) A^{(j,k)}_{t_{p-1}t_p} \right),$$

where $\hat{x}_s := \hat{X}_{0s}$, and $\mathcal{P} = \{s = t_0 < t_1 < \ldots < t_N = t\}$ is a partition on $[s,t]$. If they exist on Δ_T, we rewrite $(Y^{(1)}, Y^{(2)})$ to $\int f(\hat{X})dX$, and call it the (α, H) rough path integral of f.

Theorem 0.3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is C^{n+1}_b. Then;
(i) For any $\mathbb{X} \in \Omega_{(\alpha,H)\text{-Hld}}$, the (α, H) rough path integral $\int f(\hat{X})dX$ is well-defined, and $\int f(\hat{X})dX \in \Omega_{\alpha\text{-Hld}}([0,T], \mathbb{R}^d)$, where $\Omega_{\alpha\text{-Hld}}([0,T], \mathbb{R}^d)$ is the usual α-Hölder rough path space.
(ii) The integration map $\int : \Omega_{(\alpha,H)\text{-Hld}} \to \Omega_{\alpha\text{-Hld}}$ is locally Lipschitz continuous. More precisely, for any $M > 0$, the map $\int |_{\mathcal{E}_M}$, restricted on the set

$$\mathcal{E}_M := \left\{ \mathbb{X} \in \Omega_{(\alpha,H)\text{-Hld}} : |||\mathbb{X}|||_{\alpha,H} \leqq M \right\}$$

is Lipschitz continuous; there exist a positive constant $C > 0$ such that,

$$d_\alpha \left(\int f(\hat{V})dV, \int f(\hat{W})dW \right) \leqq C d_{(\alpha,H)}(\mathbb{V}, \mathbb{W}), \quad \mathbb{V}, \mathbb{W} \in \mathcal{E}_M,$$

where d_α is the usual metric function on $\Omega_{\alpha\text{-Hld}}([0,T], \mathbb{R}^d)$.

Proposition 0.4. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geqq 0})$ be a filtered probability space. Suppose that $X = (X^1, \ldots, X^d)$ is a d-dimensional standard Brownian motion, and by using the Itô integration, we define $\hat{X}$, $Z^{(i)}$, and $A^{(j,k)}$ as follows (note that $Z^{(0)} = X$); for $(s,t) \in \Delta_T$

$$\hat{X}_{st} := \int_0^t k(t-r)dX^1_r - \int_0^s k(s-r)dX^1_r, \quad Z^{(i)}_{st} := \frac{1}{i!} \int_s^t \left(\hat{X}_{su}\right)^i \otimes dX_u$$

$$A^{(j,k)}_{st} := \frac{1}{k!} \int_s^t \left(\hat{X}_{su}\right)^k Z^{(j)}_{su} \otimes dX_u, \quad k(r) := \frac{1}{\Gamma(H+1/2)} r^{H-1/2}$$

Then;
(i) for a.s. $\omega \in \Omega$, $\left(\hat{X}(\omega), Z^{(i)}(\omega), A^{(j,k)}(\omega)\right)_{i\in I,(j,k)\in J}$ is an (α, H) rough path.
(ii)

$$\left(\int f(\hat{X})dX \right)^{(1)}_{st} = \int_s^t f(\hat{X}_u) \otimes dX_u, \quad a.s.$$

where the left-hand-side is the first level of the (α, H) rough path integral and the right-hand-side the Itô integral.

We now discuss about the following type of RDE (in the sense of Lyon's meaning);

$$dS_t = \sigma(S_t)dZ_t, \quad Z := \int f(\hat{X})dX \in \Omega_{\alpha\text{-Hld}}([0,T], \mathbb{R}^d) \tag{0.4}$$

where $\sigma : \mathbb{R} \to \text{Mat}(1,d)$ is C^3_b.

Theorem 0.5. (i) RDE (0.4) driven by $Z = \int f(\hat{X})dX$ has the unique solution. Moreover, the solution map;

$$\Omega_{\alpha\text{-Hld}}([0,T], \mathbb{R}^d) \times \mathbb{R} \to \Omega_{\alpha\text{-Hld}}([0,T], \mathbb{R}^{d+1})$$

is locally Lipschitz continuous with respect to d_α.
(ii) The first level of the solution to RDE (0.4) is the solution to the Itô SDE (0.1).