Construction of a canonical p-energy on the Sierpiński carpet

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This talk will present the main results of [6], where a scaling limit $\mathcal{E}_p$ of discrete p-energies on a series of finite graphs approximating the planar Sierpiński carpet (SC: see Figure 1) and an associated $(1, p)$-"Sobolev" space $\mathcal{F}_p$ are established for $p > \dim_{\mathrm{ARC}}(\mathrm{SC})$. Here $\dim_{\mathrm{ARC}}(\mathrm{SC})$ is the *Ahlfors regular conformal dimension* of the SC, which is well-studied in "quasiconformal geometry" (cf. [2]). Its definition is not important to follow our results.

To give concrete descriptions, we recall the definition of the SC. Let $S = \{1, \dots, 8\}$ and let $\{F_i\}_{i\in S}$ be a family of 3^{-1}-similitudes on $\mathbb{R}^2$ illustrated in Figure 1 (each p_i in Figure 1 is the fixed point of F_i). Then the SC is a unique non-empty compact subset K of $\mathbb{R}^2$ such that $\bigcup_{i\in S} F_i(K) = K$. We always consider the Sierpiński carpet as a metric measure space (K, d, μ), where d is the Euclidean metric and μ is the normalized $\log 8/\log 3$-dimensional Hausdorff measure ($\alpha := \log 8/\log 3$ is the Hausdorff dimension of (K, d)). Let $\{G_n\}_{n\geq 1}$ be a series of finite graphs approximating SC whose vertex set is $W_n := \{w_1 \cdots w_n \mid w_i \in S \text{ for each } i \in \{1, \dots, n\}\}$ and edge set is defined by $E_n := \{(v, w) \mid v, w \in W_n \text{ and } K_v \cap K_w \neq \emptyset\}$, where $F_w := F_{w_1} \circ \cdots \circ F_{w_n}$ and $K_w := F_w(K)$ for $w = w_1 \cdots w_n \in W_n$. Then discrete p-energy $\mathcal{E}_p^{G_n}$ on G_n is defined by

$$\mathcal{E}_p^{G_n}(f) = \frac{1}{2} \sum_{(v,w)\in E_n} |M_n f(v) - M_n f(w)|^p ,$$

where $M_n f(w) := \frac{1}{\mu(K_w)} \int_{K_w} f \, d\mu$ for $f \in L^p(K, \mu)$. Our p-energy $\mathcal{E}_p$ is obtained as a scaling limit of $\mathcal{E}_p^{G_n}$. An appropriate scaling constant is determined by behaviors of $\mathcal{R}_p^{(n)}$ given by

$$\mathcal{R}_p^{(n)} := \left(\inf\left\{\mathcal{E}_p^{G_n}(M_n f) \,\middle|\, M_n f \equiv 0 \text{ on the "left" of } G_n \text{ and } M_n f \equiv 1 \text{ on the "right" of } G_n\right\}\right)^{-1}.$$

By using p-combinatorial modulus, which is a fundamental tool in "quasiconformal geometry", Bourdon and Kleiner [3] proved that there exists $\rho_p > 0$ such that $\mathcal{R}_p^{(n)} \asymp \rho_p^n$ for any $n \in \mathbb{N}$. An important fact is that $p > \dim_{\mathrm{ARC}}(K, d)$ if and only if $\rho_p > 1$ by the result of Kigami [4].

Our first main result provides $(1, p)$-"Sobolev" space $\mathcal{F}_p$ on the SC:

Theorem 1. *Assume that $p > \dim_{\mathrm{ARC}}(K, d)$. A norm space $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$, which is defined by*

$$\mathcal{F}_p := \left\{ f \in L^p(K, \mu) \,\middle|\, \sup_{n\geq 1} \rho_p^n \, \mathcal{E}_p^{G_n}(M_n f) < \infty \right\},$$

and $\|f\|_{\mathcal{F}_p} := \|f\|_{L^p} + \left(\sup_{n\geq 1} \rho_p^n \, \mathcal{E}_p^{G_n}(M_n f)\right)^{1/p}$, is a reflexive and separable Banach space. Moreover, $\mathcal{F}_p$ is continuously embedded in the Hölder space $C^{0,(\beta_p-\alpha)/p}$ on K, where $\beta_p :=$ $\log(8\rho_p)/\log 3$. Furthermore, $\mathcal{F}_p$ is dense in $C(K)$ with respect to the sup norm.

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This Hölder embedding result is powerful and relies on the assumption: $p > \dim_{\mathrm{ARC}}(K, d)$. Indeed, we can regard this embedding result as a generalization of the classical Sobolev embedding of $W^{1,p}(\mathbb{R}^N)$ for $p > N$.

Our second main result provides a "canonical" p-energy $\mathcal{E}_p$ on the SC:

Theorem 2. *Assume that $p > \dim_{\mathrm{ARC}}(K, d)$. Then there exists a functional $\mathcal{E}_p : \mathcal{F}_p \to [0, \infty)$ such that $\mathcal{E}_p(\,\cdot\,)^{1/p}$ is a semi-norm, a norm $\|\cdot\|_{\mathcal{E}_p} := \|\cdot\|_{L^p} + \mathcal{E}_p(\,\cdot\,)^{1/p}$ is equivalent to $\|\cdot\|_{\mathcal{F}_p}$, and the Banach space $(\mathcal{F}_p, \|\cdot\|_{\mathcal{E}_p})$ is uniformly convex. Furthermore, $(\mathcal{E}_p, \mathcal{F}_p)$ satisfies the following conditions:*

(a) $\mathbb{1}_K \in \mathcal{F}_p$, *and, for $f \in \mathcal{F}_p$, $\mathcal{E}_p(f) = 0$ if and only if f is constant. Furthermore, $\mathcal{E}_p(f + a\mathbb{1}_K) = \mathcal{E}_p(f)$ for any $f \in \mathcal{F}_p$ and $a \in \mathbb{R}$;*

(b) (Markov property) *for every $f \in \mathcal{F}_p$ and $\varphi : \mathbb{R} \to \mathbb{R}$ with $\mathrm{Lip}(\varphi) \leq 1$, it follows that $\varphi \circ f \in \mathcal{F}_p$ and $\mathcal{E}_p(\varphi \circ f) \leq \mathcal{E}_p(f)$;*

(c) (Symmetry) *for every $f \in \mathcal{F}_p$ and isometry $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(K) = K$, it follows that $f \circ T \in \mathcal{F}_p$ and $\mathcal{E}_p(f \circ T) = \mathcal{E}_p(f)$;*

(d) (Self-similarity) *it holds that $\mathcal{F}_p = \{f \in C(K) \mid f \circ F_i \in \mathcal{F}_p \text{ for all } i \in S\}$ and $\mathcal{E}_p(f) = \rho_p \sum_{i \in S} \mathcal{E}_p(f \circ F_i)$ for every $f \in \mathcal{F}_p$.*

I will also give an associated energy measure $\mu^p_{\langle f \rangle}(dx)$ on the SC, which plays the roles of the measure $|\nabla f|^p\ dx$ on $\mathbb{R}^N$.

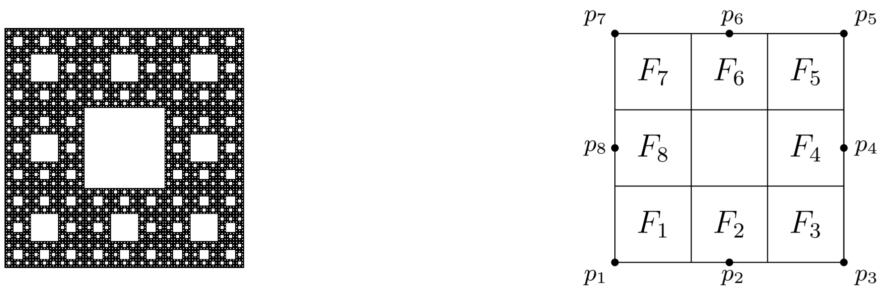

Figure 1: The planar Sierpiński carpet

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