# Renormalization of the stochastic nonlinear heat and wave equations driven by subordinate cylindrical Brownian noises

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We consider the following stochastic nonlinear heat and wave equations on two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$:

$$\mathcal{L}u = \pm u^k + \partial_t W_L \tag{1}$$

where $k \geq 2$, $\mathcal{L} = \partial_t - \Delta$ or $\partial_t^2 - \Delta$ and $W_L$ denotes a "subordinate cylindrical Brownian motion" which we define as follows: Let $W$ be a cylindrical Brownian motion on $L^2(\mathbb{T}^2)$, formally expressed by Fourier expansion

$$W(t) = \sum_{l \in \mathbb{Z}^2} \beta^l(t) e^{\sqrt{-1} l \cdot x}$$

with independent and identically distributed sequence of standard Brownian motions $\{\beta^l\}_{l \in \mathbb{Z}^2}$, and let $L$ be $\mathbb{R}_+$-valued stochastic process with nondecreasing and càdlàg sample paths. We also assume that $L$ is independent of $\{\beta^l\}_{l \in \mathbb{Z}^2}$.Then, we define $W_L$ by

$$W_L(t) := W(L(t)).$$

The main reason of considering such a time-change is that if $L$ is Lévy process, $W_L$ also becomes a Lévy process and some important Lévy processes are constructed by this "subordination" procedure.

If $L(t) = t$, $\partial_t W_L$ is nothing but a space-time white noise. Stochastic heat equation (1) with an additive space-time white noise is studied in [1]. Stochastic wave equation (1) is also considered in [2] by a similar approach. We generalize these settings and study both heat and wave equations driven by subordinate cylindrical Brownian noise.

It is expected that a solution $u$ of (1) is a distribution-valued stochastic process and the nonlinear term $u^k$ does not make sense. We overcome this difficulty by "renormalization" similarly to [1].

1. Let $\{P_N\}_{N \in \mathbb{N}}$ be mollifier and consider the equation with regularized noise $P_N \partial_t W_L$ instead of $\partial_t W_L$.

2. Then, we replace the nonlinear term $u_N^k$ by

   $$u_N^{\diamond k} := H_k(u_N; c_N)$$

   with suitable sequence $c_N$ where $H_k(x; c)$ is $k$th Hermite polynomial. (For example, $H_2(x, c) = x^2 - c, H_3(x, c) = x^3 - 3cx$.)

Thanks to this renormalization procedure, we can get a nontrivial limit $u := \lim u_N$ and we define by $u$ the solution of renormalized equation:

$$\mathcal{L}u = \pm u^{\diamond k} + \partial_t W_L \tag{2}$$

In our setting, we have to choose $c_N$ to be a $L$-measurable $\mathbb{R}_+$-valued stochastic process which diverges in some sense, while in [1] and [2], $c_N$ can be chosen as a diverging constant. Indeed, we define it by conditional expectation:

$$c_N := \mathbb{E}\left[\Psi_N(t)^2 \middle| \mathcal{F}^L\right]$$

where $\mathcal{F}^L$ is $\sigma$-algebra generated by $L$ and $\Psi_N$ is the solution of

$$\mathcal{L}\Psi_N = P_N \partial_t W_L \ . \tag{3}$$

To solve (2), we define the shifted solution $v_N \coloneqq u_N - \Psi_N$ and expand $u_N^{\diamond k}$ as

$$u_N^{\diamond k} = \sum_{l=0}^{k} \binom{k}{l} v_N^{k-l} \Psi_N^{\diamond l}$$

where $\Psi_N^{\diamond k} \coloneqq H_k(\Psi_N; c_N)$. We have the following theorem on the convergence of $\Psi_N^{\diamond k}$.

**Theorem 1.** *Let $k \in \mathbb{N}$ and let $\Psi_N$ be the solution of* (3) *with initial condition* 0*.*

1. *Let $\mathcal{L} = \partial_t - \Delta$. Then, $\Psi_N^{\diamond k}$ converges in $L^{\frac{2}{k}}([0,T]; B^{-\epsilon}_{\infty,\infty}(\mathbb{T}^2))$ as $N \to \infty$ almost surely for any $\epsilon > 0, T > 0$.*
2. *Let $\mathcal{L} = \partial_t^2 - \Delta$. Then, $\Psi_N^{\diamond k}$ converges in $C([0,T]; B^{-\epsilon}_{\infty,\infty}(\mathbb{T}^2))$ as $N \to \infty$ almost surely for any $\epsilon > 0, T > 0$.*

In the case of heat equation, we cannot expect the temporal continuity of $\Psi$ since we are dealing with the equation driven by jump-type noise. So we discuss on $L^p$-space with respect to time variable $t$. Note that in the case of $k \geq 3$, we have to consider $L^p$-space for $0 < p < 1$ since $\frac{2}{k} < 1$. We also note that it is well-known that if $L(t) = t$, $\Psi_N^{\diamond k}$ converges to some $\Psi^{\diamond k}$ in $C([0,T]; B^{-\epsilon}_{\infty,\infty})$ i.e. $\Psi^{\diamond k}$ has time-continuity. In the case of wave equation, however, we can get the continuity in time, although the noise is jump-type.

By applying Theorem 1, we can show local-in-time well-posedness of singular SPDE (2).

**Theorem 2.**

1. *Let $\mathcal{L} = \partial_t - \Delta$ and $k = 2$. Then, the renormalized heat equation* (2) *is locally well-posed.*
2. *Let $\mathcal{L} = \partial_t^2 - \Delta$. Then, for any integer $k \geq 2$, the renormalized wave equation* (2) *is locally well-posed.*

In the case of heat equation, we have not been able to deal with the case $k \geq 3$ due to the lack of time-integrability of $\Psi^{\diamond k}$ (See Theorem 1). On the other hand, in the case of wave equation, we could show local well-posedness for all $k \geq 2$. Indeed, the same fixed-point argument as in [2] is applicable to our situation in view of the time-continuity of $\Psi^{\diamond k}$.

# References


[1] G. Da Prato and A. Debussche. Strong solutions to the stochastic quantization equations. The Ann. Probab. 2003, Vol. 31, No. 4, 1900-1916.

[2] M. Gubinelli, H. Koch, and T. Oh. Renormalization of the two-dimensional stochastic nonlinear wave equations. Trans. Amer. Math. Soc. 2018, Vol. 370, No. 10, 7335-7359.