On the conformal walk dimension II: Non-attainment for some Sierpiński carpets

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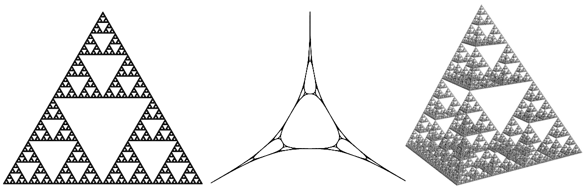

Fig. 1. Sierpiński gaskets (2-d., harm., 3-d.)

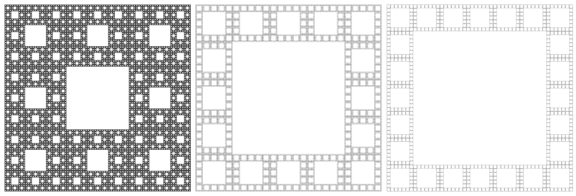

Fig. 2. Sierpiński carpets SC_ℓ ($\ell = 3, 5, 7$)

This is a continuation of the speaker's talk from 24 December 2020 on [4], which concerns the following set $\mathcal{G}_\beta(\mathcal{D})$ defined for $\beta \in (1, \infty)$ and a *metric measure Dirichlet (MMD) space* $\mathcal{D} = (K, d, m, \mathcal{E}, \mathcal{F})$, i.e., a strongly local regular symmetric Dirichlet space $(K, m, \mathcal{E}, \mathcal{F})$ over a locally compact separable metric space (K, d) such that $B_d(x, r) := \{y \in K \mid d(x, y) < r\}$ has compact closure in K for any $(x, r) \in K \times (0, \infty)$:

$$\mathcal{G}_\beta(\mathcal{D}) := \left\{ (\theta, \mu) \,\middle|\, \begin{array}{l} \theta \text{ is a metric on } K \text{ quasisymmetric to } d,\ \mu \text{ is a Radon mea-} \\ \text{sure on } K \text{ charging no set of zero } \mathcal{E}\text{-capacity and with full} \\ \mathcal{E}\text{-quasi-support, } (K, \theta, \mu, \mathcal{E}, \mathcal{F}^\mu) \text{ satisfies VD and HKE}(\beta) \end{array} \right\}. \qquad (\mathcal{G}_\beta)$$

Here we say that $(K, d, m, \mathcal{E}, \mathcal{F})$ satisfies VD if and only if $m(B_d(x, 2r)) \leq c_{\mathrm{v}} m(B_d(x, r))$ for any $(x, r) \in K \times (0, \infty)$ for some $c_{\mathrm{v}} \in (0, \infty)$, and that it satisfies HKE(β) if and only if $(K, m, \mathcal{E}, \mathcal{F})$ has a continuous heat kernel $p = p_t(x, y) : (0, \infty) \times K \times K \to [0, \infty)$ and there exist $c_1, c_2, c_3, c_4 \in (0, \infty)$ such that for any $r, t \in (0, \infty)$ and any $x, y \in K$,

$$\frac{c_1 \mathbb{1}_{[0,c_2]}\bigl(d(x, y)^\beta / t\bigr)}{m(B_d(x, t^{1/\beta}))} \leq p_t(x, y) \leq \frac{c_3 \exp\bigl(-c_4 \bigl(d(x, y)^\beta / t\bigr)^{\frac{1}{\beta - 1}}\bigr)}{m(B_d(x, t^{1/\beta}))}. \qquad \mathrm{HKE}(\beta)$$

A metric θ on K is said to be *quasisymmetric* to d ($\theta \overset{\mathrm{qs}}{\sim} d$) if and only if $\theta(x, y)/\theta(x, z) \leq \eta\bigl(d(x, y)/d(x, z)\bigr)$ for any $x, y, z \in K$ with $x \neq z$, or equivalently, for any $x \in K$ and any $r, A \in (0, \infty)$ there exists $s \in (0, \infty)$ such that $B_\theta(x, s) \subset B_d(x, r)$ and $B_d(x, Ar) \subset B_\theta(x, \eta(A)s)$, for some homeomorphism $\eta : [0, \infty) \to [0, \infty)$. Each μ as in ($\mathcal{G}_\beta$) is such that *"$\mathcal{E}$ becomes a regular Dirichlet form on $L^2(K, \mu)$ with core $\mathcal{F} \cap C_{\mathrm{c}}(K)$"*, whose domain is then denoted by $\mathcal{F}^\mu$; see [1, Corollary 5.2.10, (5.2.17) and Theorem 5.2.11] (here $C_{\mathrm{c}}(K) := \{u\colon K \to \mathbb{R} \mid u \text{ is continuous, } K \setminus u^{-1}(0) \text{ has compact closure in } K\}$).

It is relatively well known that $\mathcal{G}_\beta(\mathcal{D}) = \emptyset$ for any $\beta \in (1, 2)$ (unless K is a singleton); see [4, (1.5) and Lemma 4.7]. Our concern is whether $\mathcal{G}_\beta(\mathcal{D}) \neq \emptyset$ for $\beta = 2$, or at least for $\beta \in (2, \infty)$ arbitrarily close to 2, which is motivated by the following theorem.

Theorem 1 ([6]; see also [4, Theorem 6.30])**.** *Let $\mathcal{D}$ be the MMD space of the Brownian motion on the* 2-dimensional standard Sierpiński gasket *(Fig. 1, left). Then $\mathcal{G}_2(\mathcal{D}) \neq \emptyset$.*

More precisely, [6] constructed a concrete element of $\mathcal{G}_2(\mathcal{D})$ on the basis of the geometry of the *harmonic Sierpiński gasket* (Fig. 1, center). As an answer to the question of whether $\mathcal{G}_\beta(\mathcal{D}) \neq \emptyset$ for a general MMD space $\mathcal{D}$, in [4] we have proved the following.

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Theorem 2 ([4, Theorem 2.9]). *Let $\mathcal{D} = (K, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space with K having at least two elements. Then $\{\beta \in (1,\infty) \mid \mathcal{G}_\beta(\mathcal{D}) \neq \emptyset\}$ is $[2,\infty)$ or $(2,\infty)$ or $\emptyset$.*

Theorem 2 further raises the questions of *what $\mathcal{D}$ satisfies* $\mathcal{G}_2(\mathcal{D}) \neq \emptyset$ and *what $\mathcal{G}_2(\mathcal{D})$ looks like when $\mathcal{G}_2(\mathcal{D}) \neq \emptyset$.* In these regards, in [4] we have proved the following.

Theorem 3 ([4, Proposition 2.10]; see also [3, Section 4]). *Let $\mathcal{D} = (K, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space with K having at least two elements, and let $\mu_{\langle u \rangle}$ be the $\mathcal{E}$-energy measure of $u \in \mathcal{F}$ as defined in* [2, (3.2.14)]. *Then for any $(\theta, \mu) \in \mathcal{G}_2(\mathcal{D})$, the following hold:*

(1) *Define $d_\mu(x,y) := \sup\{u(x) - u(y) \mid u \in \mathcal{F} \cap C_{\mathrm{c}}(K),\ \mu_{\langle u \rangle} \leq \mu\}$ for each $x, y \in K$. Then $c_{\theta,\mu}^{-1} d_\mu(x,y) \leq \theta(x,y) \leq c_{\theta,\mu} d_\mu(x,y)$ for any $x, y \in K$ for some $c_{\theta,\mu} \in [1,\infty)$.*

(2) *Let A be a Borel subset of K. Then $\mu(A) = 0$ if and only if $\sup_{u \in \mathcal{F}} \mu_{\langle u \rangle}(A) = 0$.*

Theorem 4 ([4, Theorem 6.32]). *Let $N \in \mathbb{N}$ satisfy $N \geq 3$, and let $\mathcal{D}$ be the MMD space of the Brownian motion on the N*-dimensional standard Sierpiński gasket *(see Fig. 1, right, for a picture for $N = 3$). Then $\mathcal{G}_2(\mathcal{D}) = \emptyset$.*

It was left open in [4] whether $\mathcal{G}_2(\mathcal{D}) \neq \emptyset$ for the MMD space $\mathcal{D}$ of the Brownian motion on *generalized Sierpiński carpets* (see, e.g., [4, Subsection 6.4] and the references therein for their basics). We have recently been able to answer this question for a family of generalized Sierpiński carpets in $\mathbb{R}^2$ as follows, which is the main result of this talk.

Theorem 5 ([5]). *Let $\ell \in \mathbb{N} \setminus \{1\}$ be odd, and let SC_ℓ be the unique non-empty compact subset of $\mathbb{R}^2$ such that $\mathrm{SC}_\ell = \bigcup_{i \in S_\ell} f_{\ell,i}(\mathrm{SC}_\ell)$ (Fig. 2), where $f_{\ell,i} \colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f_{\ell,i}(x) := \ell^{-1} i + \ell^{-1} x$ for $i \in \mathbb{Z}^2$ and $S_\ell := \{i \in \mathbb{Z}^2 \mid f_{\ell,i}([0,1]^2) \subset [0,1]^2 \setminus (\ell^{-1}, 1-\ell^{-1})^2\}$. Then the MMD space $\mathcal{D} = (K, d, m, \mathcal{E}, \mathcal{F})$ of the Brownian motion on $K := \mathrm{SC}_\ell$, where d is the Euclidean metric and m is the uniform distribution on K, satisfies $\mathcal{G}_2(\mathcal{D}) = \emptyset$.*

Note that SC_3 (Fig. 2, left) is nothing but the 2-*dimensional standard Sierpiński carpet.*

We fix the setting of Theorem 5 in the rest of this article. The first step of the proof of Theorem 5 is to note the following theorem and proposition, which we had essentially proved in [4] for any generalized Sierpiński carpet in $\mathbb{R}^N$ with arbitrary $N \in \mathbb{N} \setminus \{1\}$.

Theorem 6 (a special case of [4, (6.70) and Theorem 6.49]). *Set $V_0 := K \setminus (0,1)^2$. If $\mathcal{G}_2(\mathcal{D}) \neq \emptyset$, then there exists $h \in \mathcal{F}$ which is $\mathcal{E}$-*harmonic *on $K \setminus V_0$, i.e., which satisfies $\mathcal{E}(h, v) = 0$ for any $v \in \mathcal{F} \cap C_{\mathrm{c}}(K)$ with $v|_{V_0} = 0$, such that $(d_{\mu_{\langle h \rangle}}, \mu_{\langle h \rangle}) \in \mathcal{G}_2(\mathcal{D})$.*

Proposition 7 (cf. [4, Proposition 6.50, Lemma 6.52 and Proof of Theorem 6.49]). *Set $\mathcal{H}_2 := \{h + \mathbb{R}\mathbf{1}_K \mid h \in \mathcal{F},\ h$ is $\mathcal{E}$-harmonic on $K \setminus V_0,\ (d_{\mu_{\langle h \rangle}}, \mu_{\langle h \rangle}) \in \mathcal{G}_2(\mathcal{D})\}$ and let $h \in \mathcal{H}_2$. Then the closure of $\{\mathcal{E}(h \circ F_{\ell,w}, h \circ F_{\ell,w})^{-1/2} h \circ F_{\ell,w}\}_{w \in \bigcup_{n=0}^\infty S_\ell^n}$ in $(\mathcal{F}/\mathbb{R}\mathbf{1}_K, \mathcal{E})$ is a compact subset of $\mathcal{H}_2$, where $F_{\ell,w} := f_{\ell,w_1} \circ \cdots \circ f_{\ell,w_n}|_K$ for $w = w_1 \ldots w_n \in S_\ell^n$.*

Theorem 5 is obtained by combining Theorem 6, Proposition 7 and the following.

Proposition 8 ([5]). *Let $h \in \mathcal{F}$ be $\mathcal{E}$-harmonic on $K \setminus V_0$ and satisfy $h|_{[0,1] \times \{j\}} = j$ for $j \in \{0,1\}$. Then $\max_{K \cap ([0,1] \times [0,\ell^{-1}])} h < \ell^{-1}$ and $d_{\mu_{\langle h \rangle}}(x,y) = 0$ for any $x, y \in [0,1] \times \{0\}$.*

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