NEW CHARACTERIZATION OF THE WEAK DISORDER PHASE OF DIRECTED POLYMERS IN BOUNDED RANDOM ENVIRONMENTS

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ABSTRACT. We show that the weak disorder phase for the directed polymer model in a random environment is characterized by the integrability of the running supremum $\sup_{n\in\mathbb{N}} W_n^\beta$ of the associated martingale $(W_n^\beta)_{n\in\mathbb{N}}$. If the environment is bounded, we also show that $(W_n^\beta)_{n\in\mathbb{N}}$ is L^p-bounded in the whole weak disorder phase, for some $p > 1$. The argument generalizes to non-negative martingales with a certain product structure.

1. Background

The *directed polymer model* was introduced in the physics literature to describe the folding of long molecule chains in a solution with random impurities. Mathematically, it is a model for random paths, called *polymers*, that are attracted or repulsed by a space-time random environment with a parameter $\beta \geq 0$, called *inverse temperature*, governing the strength of the interaction. See [2] for a recent survey of the model.

We focus on the *high temperature* phase, where it is known that the influence of the disorder disappears asymptotically and that the long-term behavior is diffusive. This weak disorder phase is characterized by whether an associated martingale, $(W_n^\beta)_{n\in\mathbb{N}}$, is uniformly integrable (UI), which is known to hold for small β if the spatial dimension is at least three.

This martingale contains a lot of information about the long-term behavior of the polymers, but (UI) is difficult to analyze in practice. Namely, (UI) means that

$$\sup_n \mathbb{E}[\varphi(W_n^\beta)] < \infty \tag{UI}$$

for some convex function φ with $\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$, but a priόri the growth of $\frac{\varphi(x)}{x}$ may be extremly slow. Much research has therefore focused on a *very high temperature* phase, characterized by L^2-boundedness of $(W_n^\beta)_{n\in\mathbb{N}}$, which is however known [1] to be a strictly stronger condition than (UI).

2. Result

We show [5, Theorem 1.1] that in the whole weak disorder phase, the martingale $(W_n^\beta)_{n\in\mathbb{N}}$ satisfies

$$\mathbb{E}\big[\sup_n W_n^\beta\big] < \infty, \tag{1}$$

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which is a strictly stronger condition than (UI). In the case of a bounded environment, we use (1) to show that $(W_n^\beta)_{n\in\mathbb{N}}$ is L^p-bounded [5, Theorem 1.1], for some $p = p(\beta) > 1$, i.e., we show that (UI) holds with $\varphi(x) = x^p$.

3. Comments

3.1. Integrability of the running maximum. We first recall an illustrative example [6, Chapter II, Exercise 3.15] for a uniformly integrable martingale that does not satisfy (1), and then discuss why the same is not possible for the martingale $(W_n^\beta)_{n\in\mathbb{N}}$ associated to the directed polymer model. The reason is that $(W_n^\beta)_{n\in\mathbb{N}}$ has a certain "product structure", i.e.,

$$W_{n+m}^\beta = W_n^\beta \widehat{W}_m^\beta, \tag{2}$$

where $\widehat{W}_m^\beta$ is a mixture of copies of W_m^β (each copy is independent of W_n^β). This structure has been noted before in the context of branching random walks or of the directed polymer on trees, but in those cases $\widehat{W}_m^\beta$ is a mixture of *independent* copies of W_m^β.

3.2. L^p-boundedness. Here, we recall [3] that (1) already implies that (UI) holds with $\varphi(x) = x\log^+(x)$ for martingales with bounded increments, which can be seen as a partial converse to Doob's maximal inequality in the case $p = 1$. The improvement to L^p-boundedness follows by a variation of this general argument, together with the product structure (2) that is specific to the directed polymer model.

3.3. Applications. Our result gives a new tool to analyse the long-term behavior of the directed polymer model in the high temperature phase. Some consequences are described in the follow-up work [4].

References

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