Itô–Föllmer calculus in infinite dimensions

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The Itô–Föllmer calculus is a pathwise approach to Itô's stochastic calculus. Föllmer [1] proved that a deterministic càdlàg path satisfies the Itô formula if it has quadratic variation along a given sequence of partitions. Föllmer's result enables one to construct the Itô type integral $\int_0^t f(X_{s-})\mathrm{d}X_s$, which is defined as the limit of non-anticipative Riemannian sums, for a nice function f and a path X that admits a quadratic variation. We call this framework the Itô–Föllmer calculus. Recently, the Itô–Föllmer calculus has seen increasing developments, receiving much attention from the viewpoint of its financial applications. In particular, it has been used to study financial trading strategies in a strictly pathwise manner.

In this talk, we consider the following differential equation in the framework of the Itô–Föllmer calculus.

$$\begin{cases} \mathrm{d}Y_t = AY_t\mathrm{d}t + \mathrm{d}X_t, & t \geq 0, \\ Y_0 = x. \end{cases} \tag{1}$$

Here, $A\colon E \to E$ is some linear operator in a Banach space and $X\colon \mathbb{R}_{\geq 0} \to E$ is a càdlàg path that corresponds to noise term. This equation is interpreted as a strictly pathwise version of a linear stochastic partial differential equation (SPDE) with additive noise. We aim to solve equation (1) explicitly in the framework of the Itô–Föllmer calculus in Banach spaces, which was developed by the author [3, 2].

To describe our results, let us define quadratic variations of a Banach space valued path. Fix a sequence $(\pi_n)_{n\in\mathbb{N}}$ of partitions on $\mathbb{R}_{\geq 0}$ such that $|\pi_n| \to 0$ as $n \to \infty$. A Banach space valued càdlàg path $X\colon \mathbb{R}_{\geq 0} \to E$ has tensor quadratic variation along (π_n) if there is a càdlàg path $[X,X]\colon \mathbb{R}_{\geq 0} \to E\widehat{\otimes}_\pi E$ of finite variation satisfying the following conditions:

(i) The sequence $\sum_{]r,s]\in\pi_n}(X_{s\wedge t} - X_{r\wedge t})^{\otimes 2}$ converges to $[X,X]_t$ for all $t \geq 0$.

(ii) The equation $\Delta[X,X]_t = \Delta X_t^{\otimes 2}$ holds for all $t \geq 0$.

Here, $E\widehat{\otimes}_\pi E$ denotes the projective tensor product of the Banach space E. Moreover, we say that X has scalar quadratic variation along (π_n) if there is an increasing càdlàg path $Q(X)\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

(i) the sequence $\sum_{]r,s]\in\pi_n}\|X_{s\wedge t} - X_{r\wedge t}\|^2$ converges to $Q(X)_t$ for all $t \geq 0$, and

(ii) the equation $\Delta Q(X)_t = \|\Delta X_t\|^2$ holds for all $t \geq 0$.

The first main result is an extension of the Itô–Föllmer formula of [3]. Given two Banach spaces E and F, let $\mathscr{L}(E,F)$ denote the space of all bounded linear operators. We define a family of seminorms (ρ_K) index by the compact subsets of E as

$$\rho_K(A) = \inf\{C > 0 \mid \|Ax\| \leq C\|x\| \text{ for all } x \in K\}.$$

We use the symbol $\mathscr{L}_{\mathrm{K}}^{\mathrm{Lip}}(E,F)$ for space $\mathscr{L}(E,F)$ with the topology induce by (ρ_K). Then, our extended Itô–Föllmer formula is stated as follows.

Theorem 1. *Let $f\colon \mathbb{R}_{\geq 0} \times E \to F$ be a continuous function satisfying the following conditions:*

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(i) *The function* $[0,\infty[\ \ni t \mapsto f(t,x) \in F$ *is differentiable for all* $x \in E$ *and* $\partial_t f(t,x)$ *is jointly continuous on* $[0,\infty[\ \times E$.

(ii) *The map* $x \mapsto f(t,x)$ *is twice Fréchet differentiable for all* $t \in [0,\infty[$ *and derivatives* $\partial_x f\colon \mathbb{R}_{\geq 0} \times E \to \mathscr{L}_{\mathrm{K}}^{\mathrm{Lip}}(E,F)$ *and* $\partial_x^2 f\colon \mathbb{R}_{\geq 0} \times E \to \mathscr{L}_{\mathrm{K}}^{\mathrm{Lip}}(E \widehat{\otimes}_\pi E, F)$ *are continuous.*

If X *has tensor and scalar quadratic variations along* (π_n)*, then*

$$f(t,X_t) - f(0,X_0) = \int_0^t \partial_s f(s,X_{s-})\mathrm{d}s + \int_0^t \partial_x f(s,X_{s-})\mathrm{d}X_s + \frac{1}{2}\int_0^t \partial_x^2 f(s,X_{s-})\mathrm{d}[X,X]_s^{\mathrm{c}} + \sum_{0<s\leq t} \{\Delta f(s,X_s) - \partial_x f(s,X_{s-})\Delta X_s\}, \tag{2}$$

where the second integral on the right hand side of (2) *is defined as the Itô–Föllmer integral.*

As an application of Theorem 1, we can solve equation (1) explicitly by means of the Itô–Föllmer integral. This result is interpreted as a pathwise version of the variation of constant formula for a linear SPDE with additive noise.

Theorem 2. *Let* $A\colon E \to E$ *be the generator of a* C_0*-semigroup* $(T_t)_{t\geq 0}$ *and* X *a càdlàg path in* $\operatorname{Dom} A$ *starting at* 0*. We suppose that* X *has tensor and scalar quadratic variations along* (π_n)*. Then there is a unique càdlàg path* Y *satisfying*

$$Y_t = x + A\int_0^t Y_s \mathrm{d}s + X_t, \qquad t \geq 0 \tag{3}$$

for every initial value $x \in E$*. The solution* Y *is represented as*

$$Y_t = T(t)x + \int_0^t \langle T(t-s), \mathrm{d}X_s\rangle, \qquad t \geq 0 \tag{4}$$

where the second term is defined as the Itô–Föllmer integral. Moreover, if $x \in \operatorname{Dom} A$*, the path of* (4) *is a classical solution to* (3)*.*

Finally, we use Theorem 2 to construct a Heath–Jarrow–Morton model of bond markets in a strictly pathwise manner. In the HJM model, forward rate dynamic is supposed to satisfy the equation

$$\mathrm{d}_t f(t,\xi) = \left(\frac{\partial}{\partial \xi} f(t,\xi) + \alpha(t,\xi)\right)\mathrm{d}t + \sigma(t,\xi)\mathrm{d}X_t.$$

With an appropriate setting of Hilbert space and a certain assumption on volatility σ, we can construct the forward rate function f in the framework of the Itô–Föllmer calculus.

References

[1] H. Föllmer. "Calcul d'Itô sans probabilités". In: *Séminaire de Probabilités XV 1979/80*. Lecture Notes in Mathematics 850. Springer Berlin Heidelberg, 1981, pp. 143–150. DOI: 10.1007/BFb0088355.

[2] Yuki Hirai. *Itô–Föllmer Calculus in Banach Space II: Transformations of Quadratic Variations*. 2021. arXiv: 2105.08262 [math.PR].

[3] Yuki Hirai. *Itô–Föllmer Calculus in Banach Spaces I: The Itô Formula*. 2021. arXiv: 2104.08138 [math.PR].