Optimal control for stochastic Volterra integral equations with delay

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This talk is based on our preprint [2]. We investigate a general class of infinite horizon optimal control problems with the state dynamics given by

$$\begin{cases} X^u(t) = \varphi(t) + \int_0^t b(t,s,X^u(s),X^u(s-\delta),u(s))\,\mathrm{d}s + \int_0^t \sigma(t,s,X^u(s),X^u(s-\delta),u(s))\,\mathrm{d}W(s), \\ \qquad\qquad t \geq 0, \\ X^u(t) = \varphi(t),\ t \in [-\delta,0]. \end{cases} \tag{1}$$

Here, $W(\cdot)$ is a one-dimensional Brownian motion, $u(\cdot)$ is a control process, $\varphi(\cdot)$ is a given adapted process called the free term, b and σ are vector-valued functions, and $\delta \geq 0$ is a constant which represents the length of the delay of the state. The state equation (1) is a stochastic Volterra integral equation (SVIE, for short) which has a "finite delay" of the form $X^u(s-\delta)$, and thus we call it a *stochastic delay Volterra integral equation* (SDVIE, for short). Our objective is to find a control process which minimizes the discounted cost functional

$$J_\lambda(u(\cdot)) := \mathbb{E}\Big[\int_0^\infty e^{-\lambda t} h(t,X^u(t),X^u(t-\delta),u(t))\,\mathrm{d}t\Big], \tag{2}$$

where h is a real-valued function called the running cost, and $\lambda > 0$ is a discount rate.

When $\delta = 0$, SDVIE (1) becomes a classical SVIE (without delay). Meanwhile, when the coefficients $b(t,s,x_1,x_2,u)$ and $\sigma(t,s,x_1,x_2,u)$ do not depend on the time-parameter t, and when the free term is of the form $\varphi(t) = \varphi(t \wedge 0)$, SDVIE (1) is reduced to a controlled stochastic delay differential equation (SDDE, for short)

$$\begin{cases} \mathrm{d}X^u(t) = b(t,X^u(t),X^u(t-\delta),u(t))\,\mathrm{d}t + \sigma(t,X^u(t),X^u(t-\delta),u(t))\,\mathrm{d}W(t),\ t \geq 0, \\ X^u(t) = \varphi(t),\ t \in [-\delta,0]. \end{cases}$$

More importantly, SDVIE (1) includes a class of controlled *fractional SDDE* of the form

$$\begin{cases} {}^{\mathrm{C}}D^\alpha_{0+}X^u(t) = b(t,X^u(t),X^u(t-\delta),u(t)) + \sigma(t,X^u(t),X^u(t-\delta),u(t))\dfrac{\mathrm{d}W(t)}{\mathrm{d}t},\ t \geq 0, \\ X^u(t) = \varphi(t),\ t \in [-\delta,0], \end{cases}$$

where ${}^{\mathrm{C}}D^\alpha_{0+}$ denotes the Caputo fractional derivative of order $\alpha \in (\frac{1}{2},1]$. Fractional differential systems are suitable tools to describe the dynamics of systems with memory effects and hereditary properties. There are many applications of fractional calculus in a variety of research fields including mathematical finance, physics, chemistry, biology, and other applied sciences. Recently, Zhang et al. [4] and Moghaddam et al. [3] studied fractional SDDEs (without control) and proved the existence and uniqueness of the solution. The analysis of stochastic control problems of fractional SDDEs is therefore an important topic, and this is a main motivation of our work.

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For the discounted control problem (1)–(2), we provide both necessary and sufficient conditions for optimality by means of Pontryagin's maximum principle. An idea here is to "lift up" the dimension of the state process so that the auxiliary state equation becomes a classical SVIE (without delay), which was investigated in our previous paper [1]. In [1], we showed that the adjoint equation, which characterizes the optimal control, of discounted control problems for classical SVIEs becomes an *infinite horizon backward stochastic Volterra integral equation* (BSVIE, for short). Applying our previous results to the auxiliary state process, we show that the adjoint equation of the original problem (1)–(2) (including delay) becomes an *infinite horizon anticipated BSVIE* (ABSVIE, for short), which is a novel class of BSVIEs whose driver depends on some "anticipated terms". The optimal control of the problem (1)–(2) is characterized by the infinite horizon ABSVIE, together with an optimality condition.

As an example of our general theory, we consider an infinite horizon linear–quadratic (LQ, for short) regulator problem for a fractional SDDE with constant coefficients. Specifically, we treat the case where the controlled state dynamics is described by a (one-dimensional) linear fractional SDDE

$$\begin{cases} {}^{\mathrm{C}}D^{\alpha}_{0+}X^u(t) = bX^u(t-\delta) + cu(t) + \sigma\dfrac{\mathrm{d}W(t)}{\mathrm{d}t},\ t \geq 0,\\ X^u(t) = x_0,\ t \in [-\delta, 0], \end{cases}$$

and the discounted cost functional is given by a quadratic functional of the state and control:

$$J_\lambda(u(\cdot)) = \frac{1}{2}\mathbb{E}\Big[\int_0^\infty e^{-\lambda t}\Big\{|X^u(t)|^2 + \frac{1}{\gamma}|u(t)|^2\Big\}\,\mathrm{d}t\Big].$$

Based on the maximum principle, we show that there exists a unique optimal control for this problem. Moreover, we obtain an explicit formula for the optimal control process of the following form:

$$\text{(optimal control)} = \text{(constant)} \times \text{(delayed optimal state)} + \text{(Gaussian process)},$$

which we call a *Gaussian state-feedback representation formula* for the optimal control. Here, the Gaussian process is a stochastic convolution of a deterministic function with respect to the Brownian motion $W(\cdot)$, and the function is determined via *linear Fredholm integral equations* depending only on the model parameters. The linear Fredoholm integral equations can be solved by using a Fredholm resolvent of the kernel, and we get the above Gaussian state-feedback representation formula.

References

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