Limit Theorems for Random Complexes

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LLN for Cubical Complex

Cubical set

- Fix $d \in \mathbb{N}$. A subset $Q \subset \mathbb{R}^d$ is said to be an elementary cube if Q has the following form:

$$Q \ = \ I_1 \times \cdots \times I_d \ ,$$

where $I_i = [a_i, a_i + 1]$ or $\{a_i\}$ for some $a_i \in \mathbb{Z}$, $1 \leq i \leq d$.
We also define the dimension of Q by

$$\dim Q \ = \ \#\{1 \leq i \leq d : |I_i| = 1\} \ .$$

- A (bounded) subset $X \subset \mathbb{R}^d$ is said to be cubical if X can be written as a union of elementary cubes:

$$X \ = \ \bigcup_{i}^{\infty} Q_i \ ,$$

for some elementary cubes Q_i, $i \in \mathbb{N}$. For any $q = 0, 1, \cdots, d$, denote the q-th Betti number of X by $\beta_q(X)$.

Random cubical set

- For two integers $0 \leq q \leq d$, denote by $\mathcal{K}_q^d$ the set of all elementary cubes in $\mathbb{R}^d$ with $\dim Q = q$. Fix $0 \leq q \leq d$. Let $\{t_Q : Q \in \mathcal{K}_q^d\}$ be a family of uniform i.i.d. r.v.'s on $[0, 1]$. For each time $t \in [0, 1]$, define the random cubical set $X(t)$ by

$$X(t) = \bigcup_{Q \in \mathcal{K}_{q-1}^d} Q \cup \bigcup_{Q \in \mathcal{K}_q^d : t_Q \leq t} Q \ ,$$

with the convention $\cup_{Q \in \mathcal{K}_{-1}^d} Q = \emptyset$.

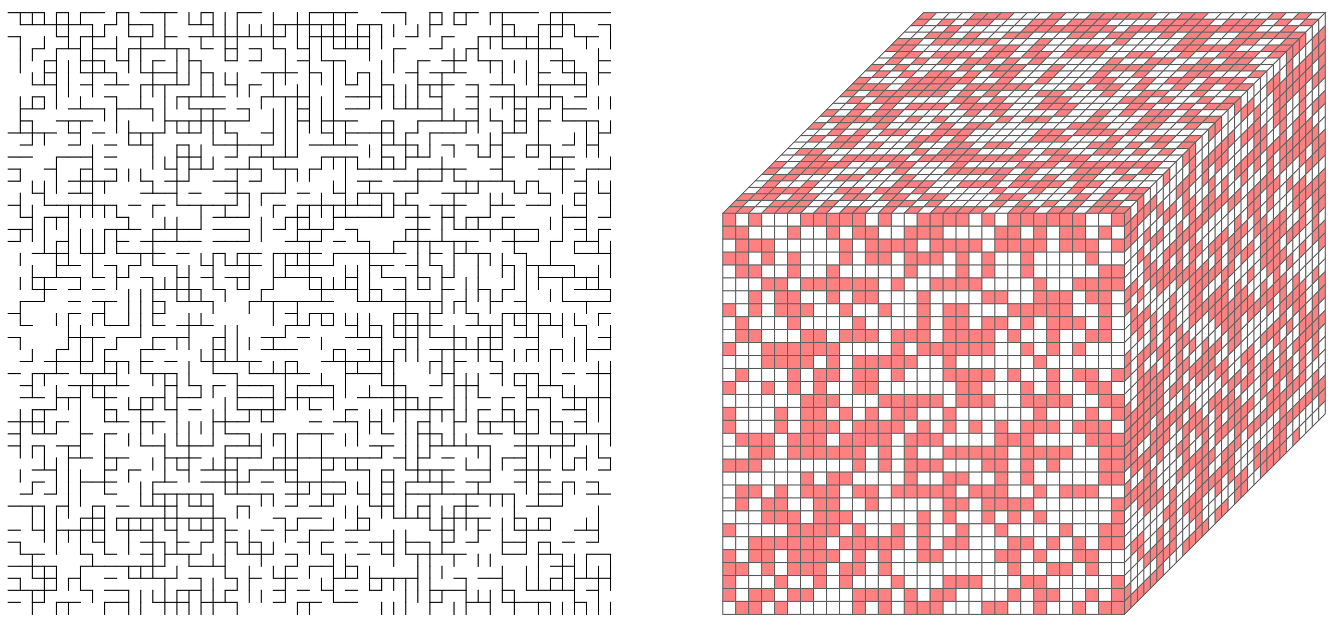

Figure: Left: $d = 2, q = 1$ without the 0th skeleton. Right: $d = 3, q = 2$.

- Fix $0 \leq q < d$. For each $n \in \mathbb{N}$ and $t \in [0, 1]$, let $\Lambda_n = [-n, n]^d$ and define $\beta_q^n(t) = \beta_q(X(t) \cap \Lambda_n)$. Define the integrated Betti number L_q^n:

$$L_q^n \ = \ \int_0^1 \beta_q^n(t) dt \ .$$

Theorem (Hiraoka-T, 18, Discrete Comput. Geom.)

There exists a function $\widehat{\beta}_q(t) : [0, 1] \to [0, \infty)$ *such that*

$$\lim_{n\to\infty} \sup_{t \in [0,1]} \left| \frac{1}{|\Lambda_n|} \beta_q^n(t) - \widehat{\beta}_q(t) \right| \ = \ 0 \ , \quad a.s.$$

In particular, we have

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \int_0^1 \beta_q^n(t) dt \ = \ \int_0^1 \widehat{\beta}_q(t) dt \ , \quad a.s.$$

Related results

- Hiraoka-Shirai, 16 $\cdots$ Correct order of $\mathbb{E}[L_q^n]$: $\mathbb{E}[L_q^n] \asymp n$.
- Werman-Wright, 16 $\cdots$ Intrinsic volumes of random cubical sets.
- Reddy-Vadlamani-Yogeshwaran, 18 $\cdots$ CLT for general functionals (including our setting essentially).
- Hiraoka-T-Miyanaga, ongoing $\cdots$ LDP for Betti numbers.

LLN for Random Čech Complex

Poisson point process and Čech complex

- Fix $d \in \mathbb{N}$. Let $\Phi = \Phi(\rho) = \{x_i\}_{i \in \mathbb{N}}$ be the Poisson point process on $\mathbb{R}^d$ with density $\rho \in [0, \infty)$:
  1. For any bounded Borel subset $A \subset \mathbb{R}^d$, the number of points in A, denoted by $\Phi(A)$, is a Poisson random variable with density $\rho|A|$.
  2. For any bounded, Borel and mutually disjoint subsets $A_1, \dots, A_n \subset \mathbb{R}^d$, $\Phi(A_1), \dots, \Phi(A_n)$ are independent.

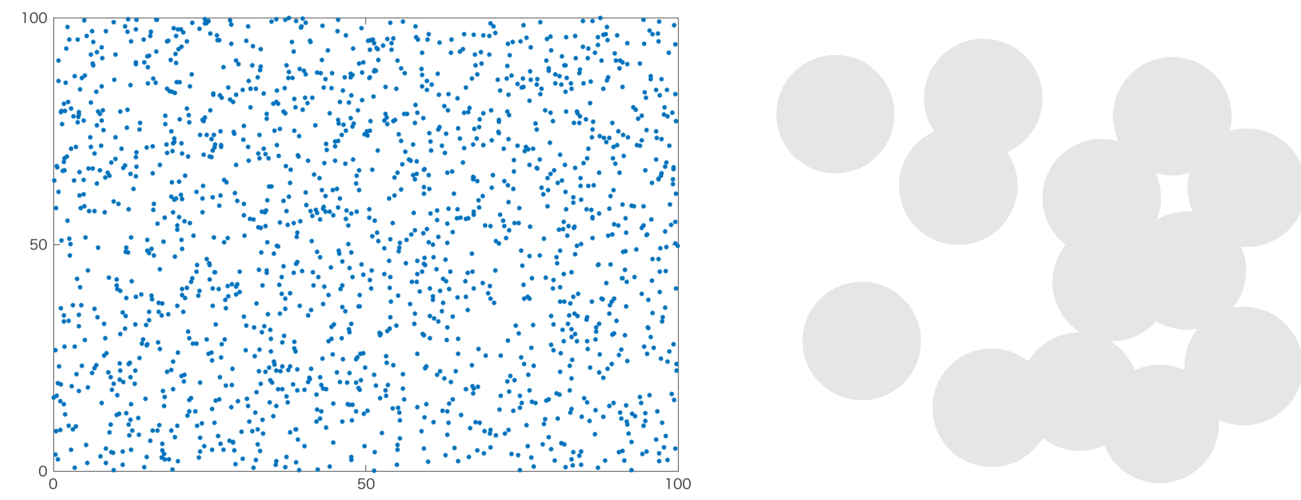

Figure: Left: Poisson point process. Right: $\beta_0 = 4, \beta_1 = 2$.

- For each $n \in \mathbb{N}$, let $\Phi_n = \Phi \cap \Lambda_n$. For each $r > 0$, we are interested in the asymptotic behavior of the Betti numbers of the Čech complex built on Φ_n with radius $r > 0$:

$$\beta_q^n(r) \ = \ \beta_q\Big(\bigcup_{x \in \Phi_n} B(x; r) \Big) \ .$$

Theorem (Yogeshwaran-Subag-Adler, '17)

There exists a deterministic constant $\bar{\beta}_q^d(\rho, r) \geq 0$ *such that*

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \beta_q^n(r) \ = \ \bar{\beta}_q^d(\rho, r) \ , \quad a.s.$$

Čech complex on manifold

- Let $\mathcal{M}$ be an m-dimensional C^1-smooth compact manifold embedded into $\mathbb{R}^d$. Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of $\mathcal{M}$-valued i.i.d. random variables with density $f : \mathcal{M} \to [0, \infty)$:

$$\mathbb{P}(X_1 \in A) \ = \ \int_A f(x) dx, \quad \forall A \subset \mathcal{M} \ .$$

Set also $\mathcal{X}_n = \{X_1, \dots, X_n\}$.

- Let $r_n \downarrow 0$ such that $n^{1/m} r_n \to r \in (0, \infty)$. Define the Betti number β_q^n by

$$\beta_q^n \ = \ \beta_q\Big(\bigcup_{x \in \mathcal{X}_n} B(x; r_n) \Big) \ .$$

Theorem (Goel-Trinh-T, 19+, J. Stat. Phys.)

Assume that $n^{1/m} r_n \to r \in (0, \infty)$ *and that* $\int_{\mathcal{M}} f(x)^p dx < \infty$, *for any* $p \in [1, \infty)$. *Then, for any integer* $0 \leq q < m$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} \beta_q^n \ = \ \int_{\mathcal{M}} \bar{\beta}_q^m(f(x), r) dx \ , \quad a.s.$$

Related results and application

- $\mathbb{E}[\beta_q^n] \asymp n$ has been shown in Bobrowski-Mukherjee, '15.
- On $\mathbb{R}^d$, when the support of f is compact, convex and satisfies

$$0 \ < \ \inf_{x \in supp(f)} f(x) \ \leq \ \sup_{x \in supp(f)} f(x) \ < \ \infty \ ,$$

a weaker version of our result has been obtained in YSA, '17.

- As an application of our result, we can also prove the LLN for the persistence diagram:

$$\lim_{n\to\infty} \frac{1}{n} \xi_q(n^{1/m} \mathcal{X}_n) \ = \ \int_{\mathcal{M}} \nu_q^m(f(x), r) dx \ , \quad a.s.$$