LDP from a hydrodynamic limit for an asymptotically degenerate particle system

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Particle system

- Each particle moves on the one-dimensional discrete torus $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z} = \{1, 2, \cdots, N\}$, $N \in \mathbb{N}$.
- Denote the number of particles at site $x \in \mathbb{T}_N$ at time t by $\eta_t(x)$.
- Let η_t be a Markov process on $\{0,1\}^{\mathbb{T}_N}$ with the generator

$$(L_N f)(\eta) = \sum_{x \in \mathbb{T}_N} c^N_{x,x+1}(\eta)\{f(\eta^{x,x+1}) - f(\eta)\},$$

where $\eta^{x,y}$ is defined by

$$\eta^{x,y}(z) = \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \\ \eta(z) & \text{if } z \neq x, y. \end{cases}$$

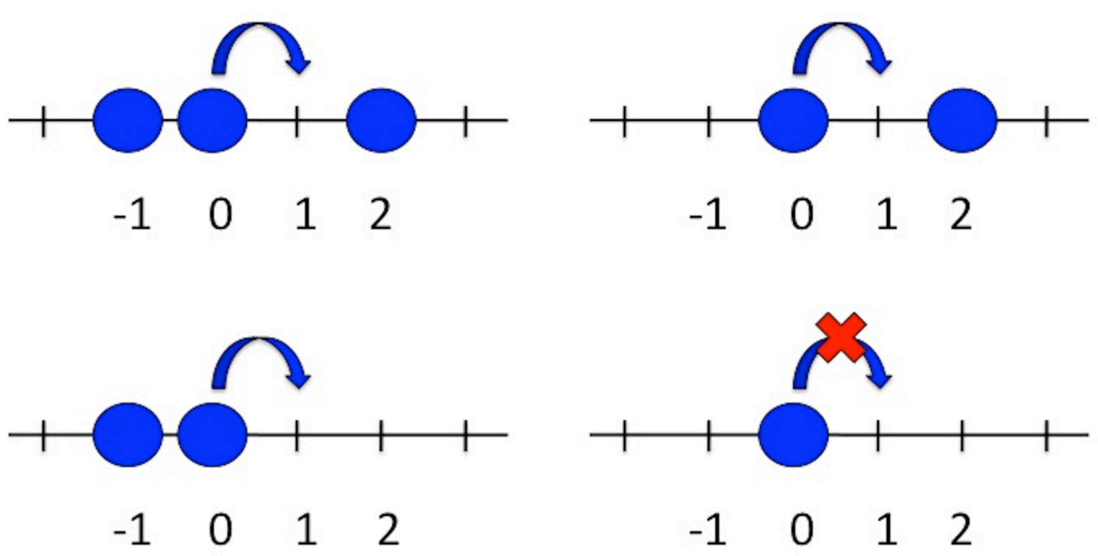
- For $j = 2, 3, \cdots$, define the local function $g_j(\eta)$ by

$$g_2(\eta) = \eta(-1) + \eta(2),$$
$$g_3(\eta) = \eta(-2)\eta(-1) + \eta(-1)\eta(2) + \eta(2)\eta(3),$$
$$g_4(\eta) = \cdots,$$

- Set $c^N_{x,x+1}(\eta) = \dfrac{1}{N} + \sum_{j=2}^n c_j g_j(\tau_x \eta)$ $(n \geq 2, c_j \geq 0)$, where

$$(\tau_x \eta)(z) = \eta(x + z).$$

Jump rates: $\eta(-1) + \eta(2)$



Diffusive scaling and Empirical measures

- Fix a continuous function $\rho_0 : \mathbb{T} \to [0,1]$ and start $\{\eta_t\}$ from the product measure $\nu_{\rho_0(\cdot)}$ with marginals

$$\nu_{\rho_0(\cdot)}(\eta; \eta(x) = 1) = \rho_0(\frac{x}{N}).$$

- Define the empirical measure π^N_t by

$$\pi^N_t(d\theta) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_{tN^2}(x) \delta_{\frac{x}{N}}(d\theta).$$

- Let $\mathbb{T}$ be the one-dimensional torus $\mathbb{R}/\mathbb{Z} = [0,1)$ and let $\mathcal{M}_+$ be the set of positive finite measures on $\mathbb{T}$.

Known result

Theorem 1 ('09, P. Gonçalves, C. Landim and C. Toninelli)

For any $t \in [0,T]$, $\{\pi^N_t(d\theta); N \geq 1\}$ converges to the deterministic measure $\rho(t,\theta)d\theta$ in probability as $N \to \infty$, where $\rho(t,\theta)$ is the unique weak solution of the PDE

$$\begin{cases} \partial_t \rho = \partial_\theta (D(\rho)\partial_\theta \rho), \\ \rho(0,\theta) = \rho_0(\theta), \end{cases}$$

where $D(\alpha) = \sum_{j=2}^n j c_j \alpha^{j-1}$. $(\alpha \in [0,1])$

Main results

- Let $D([0,T], \mathcal{M}_+(\mathbb{T}))$ be the path space of cadlag trajectories endowed with the Skorokhod topology.
- Let Q^N be the distribution on $D([0,T], \mathcal{M}_+)$ induced from the measure valued process $\{\pi^N_t\}$.
- Define the subset $\mathcal{A}$ of $D([0,T], \mathcal{M}_+(\mathbb{T}))$

$$\mathcal{A} = \{\pi(t, d\theta) = \rho(t,\theta)d\theta \mid \rho(t,\theta);\ \textit{smooth},$$
$$0 < \exists \varepsilon < \frac{1}{2}\ \textit{s.t.}\ \varepsilon \leq \rho \leq 1 - \varepsilon\}.$$

- Assume that there $\exists \varepsilon > 0$ s.t. $\varepsilon \leq \rho_0(\theta) \leq 1 - \varepsilon$.

Theorem 2 (Large deviation principle)

For each closed set $\mathcal{C}$ and each open set $\mathcal{O}$ of $D([0,T], \mathcal{M}_+(\mathbb{T}))$,

$$\limsup_{N\to\infty} \frac{1}{N} \log Q^N[\mathcal{C}] \leq -\inf_{\pi \in \mathcal{C}} I(\pi),$$
$$\liminf_{N\to\infty} \frac{1}{N} \log Q^N[\mathcal{O}] \geq -\inf_{\pi \in \mathcal{O} \cap \mathcal{A}} I(\pi).$$

Assume that $c_2 > 0$. (Technical assumption)

Theorem 3 (Large deviation principle)

The full LDP holds, that is, for each closed set $\mathcal{C}$ and each open set $\mathcal{O}$ of $D([0,T], \mathcal{M}_+(\mathbb{T}))$,

$$\limsup_{N\to\infty} \frac{1}{N} \log Q^N[\mathcal{C}] \leq -\inf_{\pi \in \mathcal{C}} I(\pi),$$
$$\liminf_{N\to\infty} \frac{1}{N} \log Q^N[\mathcal{O}] \geq -\inf_{\pi \in \mathcal{O}} I(\pi).$$

Some topics on main results

Define $\tilde{\Psi} : [0,1] \to \mathbb{R}$ by $\tilde{\Psi}(\alpha) = E_{\nu_\alpha}[\Psi]$. ($\Psi$: Local func.)

Lemma 1 (Super exponential estimate)

For each $G \in C([0,T] \times \mathbb{T})$ and each $\varepsilon > 0$, let

$$V_{N,\varepsilon}(t,\eta) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} G(t, x/N)[\tau_x \Psi(\eta) - \tilde{\Psi}(\eta^{\varepsilon N}(x))].$$

Then for any $\delta > 0$,

$$\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log P^N[|\int_0^T V_{N,\varepsilon}(t, \eta_t)dt| \geq \delta] = -\infty.$$

Remark 1

If we don't have the term $\frac{1}{N}$ in the jump rates

$$c^N_{x,x+1}(\eta) = \frac{1}{N} + \sum_{j=2}^n c_j g_j(\tau_x \eta),$$

the super exponential estimate doesn't hold!

Lemma 2 (Approximation lemma)

For each π with finite rate function ($I(\pi) < \infty$), there exists a sequence π_n in $\mathcal{A}$ converging to π in $D([0,T], \mathcal{M}_+(\mathbb{T}))$ and such that

$$\lim_{n\to\infty} I(\pi_n) = I(\pi).$$

- The important key to show the approximation lemma is control of energy of trajectories: For each $\pi \in D([0,T], \mathcal{M}^o_+)$, recall the energy $\mathcal{Q}(\pi)$;

$$\mathcal{Q}(\pi) = \sup_G \{2\int_0^T dt \int_{\mathbb{T}} d\theta\ d(\rho)\partial_\theta G - \int_0^T dt \int_{\mathbb{T}} d\theta\ \sigma(\rho) G^2\}$$
$$= \int_0^T dt \int_{\mathbb{T}} d\theta \frac{|\partial_\theta d(\rho(t,\theta))|^2}{\sigma(\rho(t,\theta))}.$$

- If $d(\rho) = \rho^2$, informally, we have

$$\mathcal{Q}(\pi) = \int_0^T dt \int_{\mathbb{T}} d\theta \frac{|2\rho\partial_\theta\rho|^2}{2\rho^2(1-\rho)} = 2\int_0^T dt \int_{\mathbb{T}} d\theta \frac{|\partial_\theta \rho|^2}{(1-\rho)}.$$

- These are informal calculations but we can verify rigorously.
- Consequently, we can deal some objects in the $\mathcal{H}^1$ sense.
- Perturbed PDE:

$$\begin{cases} \partial_t \lambda = \partial_\theta(D(\lambda)\partial_\theta \lambda) - 2\partial_\theta(\lambda(1-\lambda)D(\lambda)\partial_\theta H), \\ \lambda(0,\theta) = \gamma(\theta), \end{cases}$$

- The density of the trajectory $\pi \in D([0,T], \mathcal{M}_+(\mathbb{T}))$.

Remark 2

Without the technical assumption, we don't know differentiability of the density of the limiting trajectory π.