Angle operators and phase operators associated with 1D-harmonic oscillator

Fumio Hiroshima
Kyushu University, Fukuoka Japan

A quantum two-day meeting with green talks, green functions, and threshold behavior
Hobro, Denmark

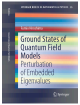
April 28, 2022

This is the joint work with NORIAKI TERANISHI

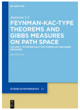
- ▶Time Operators of Harmonic Oscillators and Their Representations I (arXiv:2201.06352v4)
- ▶Time Operators of Harmonic Oscillators and Their Representations II (in preparation)

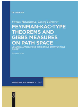
Advertisement

Ground state of quantum field models (Springer) 2019

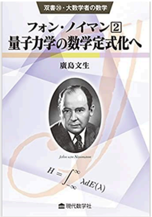


Feynman-Kac type theorem and Gibbs measures on path space vol.I and II (J.Lorinczi+V.Betz, De Gruyter) 2020





J. von Neumann I, II, III, (in Japanese) 2021



CCR

Let $[A, B]$ be the commutator of linear operators A and B defined by

$$[A, B] = AB - BA.$$

If a sa operator A in Hilbert space $\mathcal{H}$ admits a symmetric operator B satisfying CCR:

$$[A, B] = -i\mathbb{1}$$

on a non-zero subspace $D_{A,B} \subset D(AB) \cap D(BA)$, then B is called a time operator of A. We shall show several examples of time operators.

Examples: Let $\hat{h}_0 = \frac{1}{2}p^2$. $\mathrm{Spec}(\hat{h}_0) = [0, \infty)$. Let

$$\hat{T}_{AB} = \frac{1}{2}(p^{-1}q + qp^{-1}).$$

$\hat{T}_{AB}$ is called the Aharonov-Bohm operator or time of arrival operator. It holds that

$$\left[\hat{h}_0, \hat{T}_{AB}\right] = -i\mathbb{1}.$$

Question: What is a time operator of $\hat{h} = \frac{1}{2}(p^2 + q^2)$, $\mathrm{Spec}(\hat{h}) = \{n + \frac{1}{2}\}$

Domains

►When considering time operators of sa operator possessing purely discrete spectrum, we should take care of domains.
►Let $He_n = E_n e_n$ and $[H, T] = -i\mathbb{1}$. We apply e_n on both sides to result

$$(H - E_n)Te_n = -ie_n$$

and hence

$$0 = (e_n, (H - E_n)Te_n) = -i.$$

This is a contradiction. Thus we can see (1) or (2):

(1) $e_n \notin D(T)$

(2) $e_n \in D(T)$ but $Te_n \notin D(H)$

Three time operators of 1D harmonic oscillator

▶1D harmonic oscillator $\hat{h}_\varepsilon = \frac{1}{2}(p^2 + \varepsilon q^2) \quad 0 < \varepsilon \leq 1$.

1. Angle operator:

$$\hat{T}_\varepsilon = \frac{1}{2}\frac{1}{\sqrt{\varepsilon}}\left(\arctan(\sqrt{\varepsilon}p^{-1}q) + \arctan(\sqrt{\varepsilon}qp^{-1})\right).$$

The existence of a dense domain of $\hat{T}_\varepsilon$ is not trivial.

2. Galapon operator=POVM: Let P be a positive operator valued measure on a measurable space $(\Omega, \mathcal{B})$ associated to $\hat{h}_\varepsilon$. We define

$$T_G = \int_\Omega t dP_t = i\sum_{n=0}^\infty \sum_{m\neq n} \frac{(e_m, \cdot)}{m-n} e_n.$$

Then T_G becomes a time operator of $\hat{h}_\varepsilon$.

3. Phase operator: Let a and a^* be the annihilation operator and the creation operator in $L^2(\mathbb{R})$. A phase operator is formally described as

$$\hat{\phi} = \frac{i}{2}(\log a - \log a^*)$$

It is hard to define $\log a^*$ as an operator.

References

[1] Y. Aharonov and D. Bohm. Time in the quantum theory and the uncertainty relation for time and energy,. *Phys. Rev.*, 122:1649–1658, 1961.

[2] A. Arai. Generalized weak Weyl relation and decay of quantum dynamics. *Rev. Math. Phys.*, 17:1071–1109, 2005.

[3] A. Arai. Necessary and sufficient conditions for a Hamiltonian with discrete eigenvalues to have time operators. *Lett. Math. Phys.*, 87:67–80, 2009.

[4] A. Arai. *Inequivalent Representations of Canonical Commutation and Anti-Commutation Relations.* Springer, 2020.

[5] A. Arai and F. Hiroshima. Ultra-weak time operators of Schrödinger operators. *Ann. Henri Poincaré*, 18:2995–3033, 2017.

[6] A. Arai and Y. Matsuzawa. Time operators of a Hamiltonian with purely discrete spectrum. *Rev. Math. Phys.*, 20:951–978, 2008.

[7] M. Bauer. A time operator in quantum mechanics. *Ann. Phys.*, 150:1–21, 1983.

[8] C.M. Bender and G.V. Dunne. Polynomials and operator orderings. *J. Math. and Phys.*, 29:1727–1731, 1988.

[9] C.M. Bender and G.V. Dunne. Exact solutions tp operator differential equations. *Phys. Rev.D*, 40:2739–2742, 1989.

[10] C.M. Bender and G.V. Dunne. Interaction of operator differential equations. *Phys. Rev.D*, 40:3504–3511, 1989.

[11] C.M. Bender and M. Gianfreda. Matrix representation of the time operator. *J. Math. Phys*, 53:062102, 2012.

[12] C.M. Bender, R.L. Mead, and S.S. Pnsky. Continous hahn polynomials and the heisenberg algebra. *J. Math. and Phys.*, 28:509–513, 1987.

[13] F. Cannata and L. Ferrari. Canonical conjugate momentum of discrete label operators in quantum mechanics i:formalizm. *Found. Phys. Lett.*, 4:557–568, 1991.

[14] F. Cannata and L. Ferrari. Canonical conjugate momentum of discrete label operators in quantum mechanics ii:formalizm. *Found. Phys. Lett.*, 4:569–579, 1991.

[15] P. Carruthers and M.M. Niet. Phase and angle variable in quantum mechanics. *Rev. Mod. Phys.*, 40:411–440, 1968.

[16] G. Dorfmeister and J. Dorfmeister. Classification of certain pair of operators (p,q) satisfying [p,q]=-iid. *J. Funct. Anal.*, 57:301–328, 1984.

[17] B. Fuglede. On the relation $PQ - QP = -iI$. *Math. Scand.*, 20:79–88, 1967.

[18] E.A. Galapon. Pauli's theorem and quantum mechanical pairs: the consistency of bounded, self-adjoint time operator canonically conjugate to a hamiltonian with non-empty point spectrum. *Proc. R. Soc. Lond. A*, 458:451–472, 2002.

[19] E.A. Galapon. Self-adjoint time operator is the rule for discrete semi-bounded Hamiltonians. *Proc. R. Soc. Lond. A*, 458:2761–2689, 2002.

[20] E.A. Galapon, R.F. Caballar, and R. Bahague. Confined quantum time of arrivals. *Phys. Rev. Lett.*, 93:180406, 2004.

[21] E.A. Galapon, R.F. Caballar, and R. Bahague. Confined quantum time of arrivals for the vanishing potential. *Phys. Rev. A*, 72:062107, 2005.

[22] T. Goto, K. Yamaguchi, and N. Sudo. On the time opertor in quantum mechanics. *Prog. Theor. Phys.*, 66:1525–1538, 1981.

[23] T. Goto, K. Yamaguchi, and N. Sudo. On the time opertor in quantum mechanics ii. *Prog. Theor. Phys.*, 66:1915–1925., 1981.

[24] G.H. Hardy, J.L. Littlewood, and G. P'olya. *Inequalities.* Cambridge University Press, 1934.

[25] W. Heisenberg. Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Zeitschrift für Physik*, 43:172–198, 1927.

[26] F. Hiroshima and N. Teranishi. Time operators of harmonic oscillators and their representations II. in preparation.

[27] B. Leaf. Canonical operators for the simple harmonic oscillator. *J. Math. and Phys.*, 10:1980–1987, 1969.

[28] L. Mandelstam and Ig. Tamm. The uncertainty relation between energy and time in non-relativistic quantum mechanics. *J. Phys.*, 1945.

[29] M. Miyamoto. A generalized Weyl relation approach to the time operator and its connection to the survival probability. *J. Math. Phys.*, 42:1038–1052, 2001.

[30] G. Muga, R.S. Mayato, and I. Egusquiza. *Time in quantum mechanics-vol.1.* Springer, 2007.

[31] W. Pauli. *General Principles of Quantum Mechanics.* Springer, 1980.

[32] D.T. Pegg and S.M. Barnett. Phase of properties the quantized single-mode electromagnetic field. *Phys. Rev. A*, 39:1665–1675, 1989.

[33] C.R. Putnam. *Commutation Properties of Hilbert Space Operators and Related Topics.* Springer, 1967.

[34] H. Ralph, Lewis, E. Lawrence, Walter, and D. Harris, Joseph. Quantum Action-Angle Variables for the Harmonic Oscillator. *Phys. Rev. Lett.*, 77:5157–5159, 1996.

[35] H. Ralph, Lewis, E. Lawrence, Walter, and D. Harris, Joseph. A Reply to the Comment by T. B. Smith and John A. Vaccaro. *Phys. Rev. Lett.*, 80:2746–2746, 1998.

[36] M. Razavy. Quantum-machinanical time operator. *Amer. J. Phys.*, 35:955–960, 1967.

[37] D. Rosenbaum. Super hilbert space and the quantum- mechanical time operators. *J. Math. Phys.*, 10:1027–1144, 1969.

[38] K. Schmüdgen. On the heisenberg commutation relaton. i. *J. Funct. Anal.*, 50:8–49, 1983.

[39] K. Schmüdgen. On the heisenberg commutation relaton. ii. *Publ RIMS Kyoto*, (19):601–671, 1983.

[40] T.B. Smith and John.A. Vaccaro. Comment on "Quantum Action-Angle Variables for the Harmonic Oscillator". *Phys. Rev. Lett.*, 80:2745–2745, 1998.

[41] L. Susskind and J. Glogower. Quantum mechanical phase and time operator. *Physics*, 1:49–61, 1964.

[42] N. Teranishi. A note on time operators. *Lett. Math. Phys.*, 106:1259–1263, 2016.

[43] J. von Neumann. Die Eindeutigkeit der Schrődingerschen Operatoren. *Math.Ann.*, 104:570–578, 1931.

Heuristic derivation of angle operator $\hat{T}_\varepsilon$

▶Take momentum representation. $FpF^{-1} = M_k$ and $FqF^{-1} = +i\frac{d}{dk}$. Instead of notations $L^2(\mathbb{R}_k), M_k$ and $-i\frac{d}{dk}$ we denote them as $L^2(\mathbb{R}_x), q$ and p, respectively. Thus $[p, q] = -i\mathbb{1}$ also holds in the momentum representation.

▶$\hat{h}_\varepsilon$ is transformed to

$$h_\varepsilon = \frac{1}{2}(\varepsilon p^2 + q^2).$$

We shall construct symmetric operator T_ε such that

$$[h_\varepsilon, T_\varepsilon] = +i\mathbb{1}$$

in the momentum representation.

▶Let $t = q^{-1}p$ with $D(t) = \{f \in D(p) \mid pf \in D(q^{-1})\}$.

▶$[h_\varepsilon, t] = i(\mathbb{1} + \varepsilon t^2) \Longrightarrow [h_\varepsilon, f(t)] = i(\mathbb{1} + \varepsilon t^2)f'(t) \Longrightarrow f'(t) = (\mathbb{1} + \varepsilon t^2)^{-1}$

$$f(t) = \frac{1}{\sqrt{\varepsilon}} \arctan \sqrt{\varepsilon} t.$$

Symmetrizing f, we see that

$$T_\varepsilon = \frac{1}{2}\frac{1}{\sqrt{\varepsilon}}(\arctan \sqrt{\varepsilon} t + \arctan \sqrt{\varepsilon} t^*)$$

may be a time operator.

We define T_ε by using the Taylor expansion:

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad |x| < 1.$$

Note also that $\arctan x$ can be extended to a function on $\mathbb{C}$ as

$$\arctan z = \frac{i}{2} \log \frac{i+z}{i-z} \quad z \in \mathbb{C} \setminus \{i\},$$

which is a multi-valued function. Since t is unbounded and non-symmetric, it is not trivial to define

$$\arctan \sqrt{\varepsilon} t^{\#} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (\sqrt{\varepsilon} t^{\#})^{2n+1}.$$

If $\sqrt{\varepsilon} t f = i f$, then $f \notin D(\arctan \sqrt{\varepsilon} t)$.
▶It is not trivial to specify a *dense* domain D such that

$$D \subset \bigcap_{n=0}^{\infty} (D(t^n) \cap D((t^*)^n))$$

Ultra-weak time operators

► $L_0^2 = \{f \in L^2(\mathbb{R}) \mid f(-x) = f(x)\}$
► $L_1^2 = \{f \in L^2(\mathbb{R}) \mid f(-x) = -f(x)\}$.
$h_{\rm even} = h_\varepsilon\lceil_{L_0^2}$ and $h_{\rm odd} = h_\varepsilon\lceil_{L_1^2}$. Henceforth we have

$$h_\varepsilon = h_{\rm even} \oplus h_{\rm odd}.$$

Let $\mathfrak{L}_0$ and $\mathfrak{L}_1$ be

$$\mathfrak{L}_0 = {\rm LH}\{e^{-\alpha x^2/(2\sqrt{\varepsilon})} \mid \alpha \in (0,1)\} \subset L_0^2,$$
$$\mathfrak{L}_1 = {\rm LH}\{xe^{-\alpha x^2/(2\sqrt{\varepsilon})} \mid \alpha \in (0,1)\} \subset L_1^2.$$

Then $\mathfrak{L}_0 + \mathfrak{L}_1$ is dense in $L^2(\mathbb{R})$, and $\mathfrak{L}_0 \perp \mathfrak{L}_1$.

▶$S_\varepsilon^\# = S_\varepsilon, S_\varepsilon^*,\ t^\# = t, t^*$.
▶$te^{-\alpha x^2/2} = q^{-1}pe^{-\alpha x^2/2} = i\alpha e^{-\alpha x^2/2}$
▶$t^*xe^{-\alpha x^2/2} = pq^{-1}xe^{-\alpha x^2/2} = i\alpha xe^{-\alpha x^2/2}$

$$S_\varepsilon^\# = \frac{1}{\sqrt{\varepsilon}}\sum_{n=0}^\infty \frac{(-1)^n}{2n+1}(\sqrt{\varepsilon}t^\#)^{2n+1},$$

$$D(S_\varepsilon^\#) = \left\{ f \in \bigcap_{n=0}^\infty D({t^\#}^{2n+1}) \,\middle|\, \lim_{N\to\infty}\sum_{n=0}^N \frac{(-1)^n}{2n+1}(\sqrt{\varepsilon}t)^{2n+1} f \text{ exists} \right\}.$$

Formally we write $S_\varepsilon^\#$ as

$$S_\varepsilon^\# = \frac{1}{\sqrt{\varepsilon}}\arctan\sqrt{\varepsilon}t^\#.$$

Theorem (FH+Teranishi)

(1) $D(S_\varepsilon^\#) \supset \mathfrak{L}_\#$.

(2) $i(0,\infty) \subset \mathrm{Spec}_p(t^\#)$ *and* $i(0,\infty) \subset \mathrm{Spec}_p(S_\varepsilon^\#)$.

(3) $[h_\varepsilon, S_\varepsilon^\#] = i\mathbb{1}$ *on* $\mathfrak{L}_\#$.

Hierarchy of time operators (abstract theory)

►We expect $T_\varepsilon = \frac{1}{2}(S_\varepsilon + S_\varepsilon^*)$ is a time operator of h_ε. S_ε is well defined on $\mathfrak{L}_0$ and S_ε^* on $\mathfrak{L}_1$, but

$$\mathfrak{L}_0 \cap \mathfrak{L}_1 = \{0\}.$$

Instead of considering time operators, we define an ultra-weak time operator of h_ε.
►Hierarchy of classes of time operators.
{ultra-st-time} ⊂ {st-time} ⊂ {time} ⊂ {weak-time} ⊂ {ultra-weak-time}.
►Let A be a sa operator on $\mathcal{H}$ and D_1 and D_2 be non-zero subspaces of $\mathcal{H}$. A sesqui-linear form

$$\mathfrak{t}_B : D_1 \times D_2 \to \mathbb{C}, \quad D_1 \times D_2 \ni (\phi, \psi) \mapsto \mathfrak{t}_B[\phi, \psi] \in \mathbb{C}$$

with domain $D(\mathfrak{t}_B) = D_1 \times D_2$ is called an ultra-weak time operatorof A if $\exists D, \exists E \subset D_1 \cap D_2$ such that the following (1)–(3) hold: (1) $E \subset D(A) \cap D$. (2) $\mathfrak{t}_B[\phi, \psi]^* = \mathfrak{t}_B[\psi, \phi]$ for all $\phi, \psi \in D$. (3) $AE \subset D_1$ and, for all $\psi, \phi \in E$,

$$\mathfrak{t}_B[A\phi, \psi] - \mathfrak{t}_B[A\psi, \phi]^* = -i(\phi, \psi).$$

▶We define

$$\mathfrak{t}_0[\psi,\phi] = \frac{1}{2}((\psi, S_\varepsilon\phi) + (S_\varepsilon\psi,\phi)), \quad \psi,\phi \in \mathfrak{L}_0,$$

$$\mathfrak{t}_1[\psi,\phi] = \frac{1}{2}((\psi, S_\varepsilon^*\phi) + (S_\varepsilon^*\psi,\phi)), \quad \psi,\phi \in \mathfrak{L}_1.$$

$\mathfrak{t}_\varepsilon$ is defined by

$$\mathfrak{t}_\varepsilon = \mathfrak{t}_0 \oplus \mathfrak{t}_1$$

I.e., $\mathfrak{t}_\varepsilon[\psi_0 \oplus \psi_1, \phi_0 \oplus \phi_1] = \mathfrak{t}_0[\psi_0,\phi_0] + \mathfrak{t}_1[\psi_1,\phi_1]$.

Theorem (FH+Teranishi)

$\mathfrak{t}_\varepsilon$ *is an ultra-weak time operator of* h_ε *under the decomposition* $h_\varepsilon = h_{\rm even} \oplus h_{\rm odd}$.

Continuous limit

The Aharonov-Bohm operator $T_{AB} = \frac{1}{2}(t + t^*)$ can be extended to the ultra-weak time operator. Let

$$\mathfrak{t}_{AB,0}[\psi, \phi] = \frac{1}{2}\left\{(\psi, t\phi) + (t\psi, \phi)\right\} \quad \psi, \phi \in \mathfrak{M}_0,$$
$$\mathfrak{t}_{AB,1}[\psi, \phi] = \frac{1}{2}\left\{(\psi, t^*\phi) + (t^*\psi, \phi)\right\} \quad \psi, \phi \in \mathfrak{M}_1.$$

Define $\mathfrak{t}_{AB}$ by

$$\mathfrak{t}_{AB} = \mathfrak{t}_{AB,0} \oplus \mathfrak{t}_{AB,1}.$$

We can see that $\mathfrak{t}_{AB}$ is an ultra-weak time operator of $\frac{1}{2}q^2$.

Theorem (FH+Teranishi)

$$\lim_{\varepsilon \to 0} \mathfrak{t}_\varepsilon[\psi, \phi] = \mathfrak{t}_{AB}[\psi, \phi].$$

Matrix representations for $\alpha \in (0,1)$

We set $\varepsilon = 1$, and $\mathfrak{t}_{\varepsilon=1} = \mathfrak{t}$, $S^{\#}_{\varepsilon=1} = S^{\#}$ and $h_{\varepsilon=1} = h$. We also set

$$\xi_\alpha = e^{-\alpha x^2/2}.$$

►We want to see the function K_{ab} such that

$$\mathfrak{t}[x^a\xi_\alpha, x^b\xi_\alpha] = (\xi_\alpha, K_{ab}\xi_\alpha), \quad a, b \in \mathbb{N} \cup \{0\}.$$

Let us set

$$t_\alpha = 2\left(\frac{x^2}{2} - \frac{d}{d\alpha}\right).$$

Theorem (FH+Teranishi)

Suppose that $\alpha \in (0,1)$. Let ρ be a polynomial. Then

$$S\rho(x^2)\xi_\alpha = \frac{i}{2}\left(\rho(t_\alpha)\log\frac{1+\alpha}{1-\alpha}\right)\xi_\alpha,$$

$$S^*\rho(x^2)x\xi_\alpha = \frac{i}{2}\left(\rho(t_\alpha)\log\frac{1+\alpha}{1-\alpha}\right)x\xi_\alpha.$$

Together with them we have the matrix representation of $\mathfrak{t}$. Let

$$\mathfrak{K}_\alpha = \mathrm{LH}\{x^n e^{-\alpha x^2/2} \mid n \in \mathbb{N} \cup \{0\}\}.$$

We can see $\mathfrak{t}[f_a, f_b]$ for $f_a, f_b \in \mathfrak{K}_\alpha$ in the corollary below.

Corollary

Fix $\alpha \in (0,1)$. Let $f_a = x^a \xi_\alpha$ and $f_b = x^b \xi_\alpha$. Then $\mathfrak{t}[f_a, f_b]$ is given by

$$\begin{cases} -\frac{i}{4}\left(\xi_\alpha, \left\{(t_\alpha^n x^{2m} - x^{2n} t_\alpha^m) \log \frac{1+\alpha}{1-\alpha}\right\} \xi_\alpha\right) & a = 2n, b = 2m \\ -\frac{i}{4}\left(\xi_\alpha, \left\{(t_\alpha^n x^{2m+2} - x^{2n+2} t_\alpha^m) \log \frac{1+\alpha}{1-\alpha}\right\} \xi_\alpha\right) & a = 2n+1, b = 2m+1 \\ 0 & \textit{otherwise}. \end{cases}$$

We discuss the case of $\alpha = 1$. Let

$$\mathfrak{K} = \mathrm{LH}\{x^n e^{-x^2/2} \mid n \in \mathbb{N} \cup \{0\}\}.$$

Theorem (FH+Teranishi)

$\mathfrak{K} \cap D(S^\#) = \{0\}$. *In particular, let e_n be an ev of h, then $e_n \notin D(S^\#)$.*

Galapon operator

$$T_G f = i \sum_n \sum_{m \neq n} \frac{(e_m, f)}{m-n} e_n.$$

We define the unbounded operator P_0 by

$$P_0 = \lim_{N\to\infty} \frac{1}{2\pi} \left(\sum_{n=0}^{N} e_n, \cdot \right) \sum_{n=0}^{N} e_n.$$

Proposition

$[h, T_G] = -i(2\pi P_0 - \mathbb{1})$ *and* $[h, T_G] = i\mathbb{1}$ *on* $\mathrm{LH}\{e_n - e_m \mid n \neq m\}$.

We can also define the sesqui-linear form associated with T_G by

$$\mathfrak{t}_G[\phi, \psi] = \frac{1}{2}\{(\phi, T_G\psi) + (T_G\phi, \psi)\}.$$

Theorem (angle operator $\neq$ Galapon operator)

$\mathfrak{t} \neq \mathfrak{t}_G$.

Proof: $\mathfrak{t}_G$ is bounded, but $\mathfrak{t}$ is unbounded.

Phase operator

$a = p + iq$ and $a^* = p - iq$. Then $[a, a^*] = \mathbb{1}$. Let $N = a^* a$. Phase operator $\hat{\phi}$ satisfies $[N, \hat{\phi}] = i\mathbb{1}$. Heuristically we have

$$[N, f(a)] = -f'(a)a.$$

Setting $-f'(a)a = +i\mathbb{1}$, we implicitly yield that $f(a) = -i \log a$.
▶We formally have

$$\hat{\phi} = -\frac{i}{2}(\log a - \log a^*).$$

▶Let A be a linear operator in $L^2(\mathbb{R})$. We define $\log A$ by

$$\log A = -\sum_{n=1}^{\infty} \frac{1}{n}(\mathbb{1} - A)^n$$

with the domain

$$D(\log A) = \left\{ f \in L^2(\mathbb{R}) \;\middle|\; \sum_{n=1}^{\infty} \frac{1}{n}(\mathbb{1} - A)^n f \text{ strongly converges} \right\}.$$

▶$L^2(\mathbb{R}) = \oplus_{n=0}^{\infty} L_n$, where

$$L_n = \mathrm{LH}\left\{ \frac{1}{\sqrt{n!}} (\prod^n a^*) \Omega \right\},$$

where $\Omega(x) = \pi^{-1/4} e^{-x^2/2}$ and $\| \frac{1}{\sqrt{n!}} (\prod^n a^*) \Omega \| = 1$.

$$a^* L_n \to L_{n+1}, \quad a : L_n \to L_{n-1}$$

Let $\mathfrak{D}_{finite}$ be the finite particle subspace defined by

$$\mathfrak{D}_{finite} = \mathrm{LH}\left\{ f = \sum_{n=0}^{\infty} \frac{c_n}{\sqrt{n!}} \prod^n a^* \Omega \,\middle|\, n = 0 \ for \ n \geq \exists m \right\}.$$

Lemma

$\mathfrak{D}_{finite} \subset D(\log a)$ and $\mathfrak{D}_{finite} \cap D(\log a^*) = \{0\}$. In particular $D(\hat{\phi}) \cap \mathfrak{D}_{finite} = \{0\}$.

From Lemma we can see that $\hat{\phi}$ is not well defined on $\mathfrak{D}_{finite}$. This fact is fatal to consider $\hat{\phi}$.

Angle operator and phase operator

►Another candidate of a time operator is formally given by

$$\hat{\phi}_* = -\frac{i}{2}(\log a^{*\,-1}a + \log aa^{*\,-1}).$$

Note that formally $\left\{\frac{i}{2}\log a^{*\,-1}a\right\}^* = \frac{i}{2}\log aa^{*\,-1}$.
►Let

$$\mathfrak{D} = \left\{ f = \sum_{n=0}^{\infty} \frac{c_n}{\sqrt{n!}} \prod^{n} a^*\Omega \;\middle|\; c_0 = 0, \sum_{n=1}^{\infty} \frac{c_n^2}{n} < \infty \right\}.$$

The operator $a^{*\,-1}$ is defined by

$$a^{*\,-1}f = \sum_{n=1}^{\infty} \frac{c_n}{\sqrt{n!}} \prod^{n-1} a^*\Omega, \quad D(a^{*\,-1}) = \mathfrak{D}.$$

Let $\{e_n\}_n$ be the set of normalized eigenvectors of N which satisfies that $Ne_n = ne_n$. $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a unitary operator defined by

$$Ue_n = (n!)^{-1/2} {a^*}^n \Omega, \quad n \geq 0.$$

We define subspaces of $L^2(\mathbb{R})$ by

$$\mathfrak{L}_0 = \mathrm{LH}\{e^{-\alpha x^2/2} \mid \alpha \in (0,1)\},$$
$$\mathfrak{L}_1 = \mathrm{LH}\{x e^{-\alpha x^2/2} \mid \alpha \in (0,1)\}.$$

Lemma (FH+Teranishi)

$$U \arctan(q^{-1}p) U^* = -\frac{i}{2} \log\left({a^*}^{-1} a\right) \quad \text{on } U\mathfrak{L}_0,$$
$$U \arctan(pq^{-1}) U^* = -\frac{i}{2} \log\left(a {a^*}^{-1}\right) \quad \text{on } U\mathfrak{L}_1.$$

Let

$$S = -\frac{i}{2}\log(a^{*-1}a), \quad S^* = -\frac{i}{2}\log(aa^{*-1}).$$

We define

$$\mathfrak{t}_0[\phi,\psi] = \frac{1}{2}\left\{(S\phi,\psi)+(\phi,S\psi)\right\}, \quad \phi,\psi \in U\mathfrak{L}_0,$$
$$\mathfrak{t}_1[\phi,\psi] = \frac{1}{2}\left\{(S^*\phi,\psi)+(\psi,S^*\phi)\right\}, \quad \phi,\psi \in U\mathfrak{L}_1.$$

We define

$$\mathfrak{t}_* = \mathfrak{t}_0 \oplus \mathfrak{t}_1.$$

Moreover the ultra-weak time operator associated with angle operator $T = \frac{1}{2}(\arctan t + \arctan t^*)$ is denoted by $\mathfrak{t}$.

Theorem (FH+Teranishi)

$\mathfrak{t}$ *and* $\mathfrak{t}_*$ *are unitary equivalent, i.e.,* $\mathfrak{t}[\phi,\psi] = \mathfrak{t}_*[U\phi, U\psi]$.

Shift operator

► $L^2(\mathbb{R}) \cong \ell^2(\mathbb{N}^\times)$.
Let L be the left-shift and the adjoint L^* the right-shift. Let $f \in \ell^2(\mathbb{N}^\times)$. It is defined by

$$(Lf)^{(n)} = \begin{cases} f^{(n-1)} & n \geq 1 \\ 0 & n = 0 \end{cases}$$

$$(L^* f)^{(n)} = f^{(n+1)}.$$

We can see that

$$LL^* = \mathbb{1}, \quad L^* L = \mathbb{1} - P_\Omega,$$

where P_Ω is the projection to 1D subspace spanned by Ω.
In terms of the shift $L^\#$, we obtain that

$$a = L\sqrt{N} = \sqrt{N + \mathbb{1}}L,$$

$$a^* = L^* \sqrt{N + \mathbb{1}} = \sqrt{N} L^*.$$

Note that $N = a^* a$ and $N + \mathbb{1} = aa^*$.

Galapon operators and shift operators

Let

$$L_G = i\left\{\log(\mathbb{1} - L) - \log(\mathbb{1} - L^*)\right\}.$$

Lemma

(1) $\mathfrak{D}_{finite} \subset D(\log(\mathbb{1} - L^{\#}))$*. In particular* L_G *is well defined on* $\mathfrak{D}_{finite}$*.*
(2) $[N, L_G] = -i\mathbb{1}$ *on* $\mathrm{Ran}\big((\mathbb{1} - L)L^*\big) \cap D(NL_G) \cap D(L_G N)$*.*

Let us remind you that

$$T_G f = i\sum_{n=0}^{\infty}\left(\sum_{m\neq n}\frac{(e_m, f)}{n - m}e_n\right).$$

T_G is bounded and $[N, T_G] = i\mathbb{1}$ on $\mathrm{LH}\{e_n - e_m \mid n \neq m\}$.
▶$\mathrm{LH}\{e_n - e_m\} \subseteq \mathrm{Ran}\big((\mathbb{1} - L)L^*\big) \cap D(NL_G) \cap D(L_G N)$.

Theorem (FH+Teranishi)

$T_G = L_G$ *on* $D\big(\log(\mathbb{1} - L)\big) \cap D\big(\log(\mathbb{1} - L^*)\big)$*. In particular* L_G *has the bounded operator extension.*

Concluding remarks

In physics it is formally treated that $[h, A] = +i\mathbb{1}$ for

$$A = T, T_G, \hat{\phi},$$

where $T = \frac{1}{2}(\arctan q^{-1}p + \arctan pq^{-1})$, $T_G = i\sum_n \sum_{m\neq n} \frac{(e_m,\cdot)}{m-n} e_n$
$\hat{\phi} = \frac{i}{2}(\log a - \log a^*)$. We made relationships among them clear.

(1) $T \neq T_G$.

(2) If T is defined in the sense of sesqui-linear form $\mathfrak{t}$, then the domain of $\mathfrak{t}$ is dense and $\mathfrak{t}[h\phi, \psi] - \mathfrak{t}[h\psi, \phi]^* = -i(\phi, \psi)$ hols on a dense subspace.

(3) The continuous limit of T_ε is $T_{AB} = \frac{1}{2}(q^{-1}p + pq^{-1})$.

(4a) A matrix representation of $\mathfrak{t}$ is given for $\alpha \in (0, 1)$.

(4b) It can be extended to $i\alpha \in \mathbb{H} \setminus \{i\}$.

(5) $D(\hat{\phi}) \cap \mathfrak{D}_{finite} = \{0\}$.

(6) $T \cong \frac{i}{2}(\log a^{*-1}a + \log aa^{*-1})$.

(7) $T_G = i\{\log(\mathbb{1} - L) - \log(\mathbb{1} - L^*)\}$ holds true for shift operator L.

(8) We can construct time operators of the form $c(\log f(L) - \log f(L^*))$.